Problems in Quantum Groups

Try to solve at least 5 of the problems listed below.

Hopf Algebras

Problem 1 *Denote by C an associative algebra generated over $\mathbb{C}$ by elements t and x satisfying relations $t^2 = 1,\ x^2 = 0,\ tx + xt = 0$. As a vector space, C is a 4-dimensional vector space with a base $(1, t, x, tx)$. Define two linear mappings $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow \mathbb{C}$ by the formulas:*

$$\Delta(1) = 1 \otimes 1, \quad \Delta(t) = t \otimes t, \quad \Delta(x) = x \otimes 1 + 1 \otimes x, \quad \Delta(tx) = tx \otimes t + t \otimes tx,$$

$$\varepsilon(1) = 1, \qquad \varepsilon(t) = 1, \qquad \varepsilon(x) = 0, \qquad \varepsilon(tx) = 0.$$

(1.) Whether (C, Δ, ε) is a bialgebra?

(2.) Can you find an antipode for it?

Problem 2 *Let (B, Δ, ε) be a bialgebra, $S : B \longrightarrow B$ a $\mathbb{C}$ linear map satisfying $(S \otimes \mathrm{id}_B) \circ \Delta = (\mathrm{id}_B \otimes S) \circ \Delta = \eta \circ \varepsilon$.*

(1.) Prove that S is an antialgebra map.

(2.) Prove that S is an anticoalgebra map.

Problem 3 *Let G be a finite group, $H = \mathbb{C}[G]$ the group algebra of G.* Prove that the set of all grouplike elements of B coincides exactly with G.

Problem 4 *Let $(A, \mu, \Delta, \iota, \varepsilon, S)$ be a Hopf algebra. Suppose S' is a linear isomorphism $A \longrightarrow A$ such that*

$$\mu \circ (\mathrm{id} \otimes S') \circ \Delta = \mu \circ (S' \otimes \mathrm{id}) \circ \Delta = \iota \circ \varepsilon,$$

i.e. $(A, \mu, \Delta, \iota, \varepsilon, S')$ is a Hopf algebra. Prove that $S' = S$.

Problem 5 *Let $(A, \mu, \Delta, \iota, \varepsilon, S)$ be a commutative or cocommutative Hopf algebra.* Prove that $S^2 = \mathrm{id}$.

Dual Hopf algebras

Recall that a linear map $f : V \longrightarrow W$ determins a new linear map $f^* : W^* \longrightarrow V^*$ by $f^*(\psi) := \psi \circ f$ (the composition of maps), here $\psi \in W^*$. Recall also that for any two vector spaces U and V one has $(U \otimes V)^* \subset U^* \otimes V^*$. If the spaces U and V are of finite dimension, then $(U \otimes V)^* = U^* \otimes V^*$.

Problem 6 *Let $(H, \mu, \iota, \Delta, \varepsilon, S)$ be a finite dimensional Hopf algebra over $\mathbb{C}$, H^* the dual vector space of H.* Prove that $(H^*, \Delta^*, \varepsilon^*, \mu^*, \iota^*, S^*)$ is again a Hopf algebra.

Problem 7 *Let C^* be the dual vector space to the associative algebra C from Problem 1. Define τ, $\xi \in C^*$ by the following relations:*

$$\begin{array}{llll} \tau(1) = 1 \otimes 1, & \tau(t) = -1, & \tau(x) = 0, & \tau(tx) = 0, \\ \xi(1) = 0, & \xi(t) = 0, & \xi(x) = 1, & \xi(tx) = -1. \end{array}$$

(1.) Prove that the relations $\tau^2 = 1$, $\xi^2 = 0$ and $\tau\xi + \xi\tau = 0$ are hold in C^*. Give explicit formulas for the coproduct in C^*.
(2.) Prove that a linear map $\chi : C \longrightarrow C^*$ defined by

$$\chi(1) = \varepsilon, \quad \chi(t) = \tau, \quad \chi(x) = \xi, \quad \chi(tx) = \tau$$

is a bialgebra isomorphism.

Problem 8 *Let A be an associative algebra, $S : A \longrightarrow A$ an antialgebra map (i.e. satisfying $S(ab) = S(b)S(a)$), V a left A module. For $a \in A$, $\varphi \in V^*$ and $v \in V$, set*

$$(a\varphi)(v) := \varphi(S(a)v).$$

Prove that by this formula a left A module structure on V^* is defined.

Problem 9 *Let G be a finite group, $\mathbb{C}[G]$ its group algebra and $\mathcal{F}(G)$ the algebra of polynomial $\mathbb{C}$ valued functions on G.* Prove that $\mathcal{F}(G)$ is the dual Hopf algebra to $\mathbb{C}[G]$.

Drinfeld's Double

Problem 10 (Smash product) *Let H be a Hopf algebra, A a left module algebra over H. Introduce in the tensor product $A \otimes H$ of $\mathbb{C}$ vector spaces a multiplication*

$$(a \otimes x)(b \otimes y) = \big(a(x_{(1)} \triangleright b) \otimes (x_{(2)} \triangleright y)\big)$$

where $a, b \in A$, $x, y \in H$, $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, and H acts upon itself by the adjoint action: $x \triangleright y = S(x)yx$. Prove that this multiplication turns $A \otimes H$ into an associative H algebra. This algebra is usually denoted by $A \rtimes H$ and is called smash (or semi-direct) product of A and H.

Problem 11 (Drinfeld's double) *Let H be a Hopf algebra. Set $\mathfrak{D}(H) = H^* \rtimes H$, where H acts on H^* by the coadjoint action.* Prove that the multiplication in $\mathfrak{D}(H)$ is H equivariant.

In other words, $\mathfrak{D}(H)$ is a comodule algebra over H.

Quasitriangular Hopf algebras

Problem 12 *Denote by $\{e_1, \dots, e_n\}$ the canonical base of the vector space $\mathbb{C}^n$, and define a linear operator $R \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ by the formula:*

$$R(e_i \otimes e_j) = \begin{cases} e_j \otimes e_i + (q - q^{-1})\ e_i \otimes e_j, & \text{if} \quad i < j; \\ q\ e_i \otimes e_j, & \text{if} \quad i = j; \\ e_j \otimes e_i, & \text{if} \quad i > j. \end{cases}$$

(1.) Prove that R satisfies the following version of the Yang—Baxter equation:

$$R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23},$$

where $R_{12} := R \otimes \mathrm{id} \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n)$.
(2.) Prove that R satisfies the Hecke equation (in $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$:

$$(R - q\,\mathrm{id})(R + q^{-1}\,\mathrm{id}) = 0.$$

(3.) Prove that R is invertible (*Hint:* Use the Hecke Equation).

We call a Hopf algebra *quasitriangular if it is endowed with an R-matrix R and the following condition is satisfied:*

$$\Delta^{\mathrm{op}}(x) = R \cdot \Delta(x) \cdot R^{-1},$$

for any $x \in H$.

Problem 13 *Let (H, R) be a quasitriangular Hopf algebra such that the following* Drinfeld relation are satisfied:

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}, \quad (\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}.$$

Prove the following relations:
(2.) $(S \otimes \mathrm{id})R = R^{-1}$

Note that $(\mathrm{id} \otimes S)R \neq R^{-1}$

Problem 14 *Denote $Q = R_{21}R$ and set $\kappa(\varphi) = \varphi(Q_{(1)})Q_{(2)}$ for any $\varphi \in H^*$.* Prove that the κ is a bialgebra map $(H^*)^{\mathrm{op}} \longrightarrow H$.