Algebras and Representation Theory (2020) 23:1827–1848
https://doi.org/10.1007/s10468-019-09913-4

Quantum Exceptional Group G_2 and its Semisimple Conjugacy Classes

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Received: 14 May 2018 / Accepted: 9 July 2019 / Published online: 6 August 2019

Abstract

We construct quantization of semisimple conjugacy classes of the exceptional group $G = G_2$ along with and by means of their representations on highest weight modules over the quantum group $U_q(\mathfrak{g})$. With every point t of a fixed maximal torus we associate a highest weight module M_t over $U_q(\mathfrak{g})$ and realize the quantized polynomial algebra of the class of t by linear operators on M_t. Quantizations corresponding to points of the same orbit of the Weyl group are isomorphic.

Keywords Quantum groups · Conjugacy classes · Quantization

Mathematics Subject Classification 81R50 · 81R60 · 17B37

1 Introduction

Exceptional Lie groups occupy a special position in mathematics among simple groups and find important applications in theoretical physics in connection with string theories, supergravity, and grand unification, [11, 20]. At the same time their quantum analogs are the least studied compared to other quantum groups. In this paper we focus on the smallest exceptional group, $G = G_2$ and construct quantization of its semisimple conjugacy classes via an operator algebra realization.

Recall the definition of an equivariant quantization of a homogeneous G-space O. This is a $\mathbb{C}(q)$-algebra $\mathbb{C}_q[O]$ which is a deformation of the algebra $\mathbb{C}[O]$ of polynomial functions

Presented by: Iain Gordon

In memoriam Petr Kulish

This study (Andrey Mudrov) is supported in part by the RFBR grant 15-01-03148.

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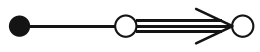
on O that supports an action of the quantum group $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. This action satisfies the quantum Leibniz rule $x.(ab) = (x_{(1)}.a)(x_{(2)}.b)$ for any $a, b \in \mathbb{C}_q[O]$, $x \in U_q(\mathfrak{g})$, where $x_{(1)} \otimes x_{(2)}$ is the Heyneman-Sweedler notation for the coproduct of x. In the classical limit $q \to 1$, this action goes over to the classical action of $U(\mathfrak{g})$ on $\mathbb{C}[O]$.

In the present work, we consider $G = G_2$ as an algebraic subgroup of $GL(\mathbb{C}^7)$, where $\mathbb{C}^7$ has the structure of the minimal fundamental representation. The triangular decomposition of $GL(\mathbb{C}^7)$ induces a triangular decomposition of G with a maximal torus $T \subset G$ represented by diagonal matrices. We denote by $O_t \subset G$ the conjugacy class of an element $t \in T$.

It is known that stabilizers of semisimple conjugacy classes can be described with the help of the affine Dynkin diagram of the group, by erasing one or more nodes. They are divided in two types: the ones whose stabilizer K is a Levi subgroup, and the others which we call pseudo-Levi subgroups. In our case ($G = G_2$), the affine Dynkin diagram for $\mathfrak{g}$ is



It turns out that a Levi K is described with at most one node, while all pseudo-Levi K correspond to exactly two nodes of the affine diagram.

We get our quantization $\mathbb{C}_q[O_t]$ of the coordinate algebra $\mathbb{C}[O_t]$ from a quantization $\mathbb{C}_q[G]$ of the coordinate algebra $\mathbb{C}[G]$ of the group G, as a quotient by an invariant ideal $N_q(O_t)$. Thus we 'deform' the classical situation, where $\mathbb{C}[O_t]$ is the polynomial ring of an affine variety, i.e. is a quotient of the algebra $\mathbb{C}[G]$. The quantization $\mathbb{C}_q[G]$ is realized as a subalgebra in $U_q(\mathfrak{g})$ generated by entries of a matrix $\mathcal{Q}$. This matrix is constructed from the universal R-matrix $\mathcal{R}$ of $U_q(\mathfrak{g})$ in the following way. Denote the above mentioned minimal representations of $U_q(\mathfrak{g})$ on $\mathbb{C}^7$ by π and set $\mathcal{Q} := (\pi \otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R})$. This is an element of $\mathrm{End}(\mathbb{C}^7) \otimes U_q(\mathfrak{g})$ and is considered as a 7×7-matrix with entries in $U_q(\mathfrak{g})$. Those entries are the generators of $\mathbb{C}_q[G] \subset U_q(\mathfrak{g})$, This subalgebra is ad-invariant due to transformation properties of $\mathcal{Q}$. In particular, the matrix $\mathcal{Q}$ satisfies the Reflection Equation $\mathcal{Q}_2 R_{21} \mathcal{Q}_1 R_{12} = R_{12} \mathcal{Q}_1 R_{21} \mathcal{Q}_2$ [6, 13], as well as some more relations, whose exact form is not important for this study.

Besides presenting the algebra $\mathbb{C}_q[O_t]$ as a quotient of $\mathbb{C}_q[G]$, we realize it by linear operators on a vector space M. As $\mathbb{C}_q[O_t]$ is a subalgebra in $U_q(\mathfrak{g})$, we take for M a $U_q(\mathfrak{g})$-module associated with a point in $T \cap O_t$. Then the ideal $N_q(O_t)$ is the annihilator of M, and the image of $\mathbb{C}_q[G]$ in $\mathrm{End}(M)$ is the quantization $\mathbb{C}_q[O_t]$. It is automatically equivariant by the construction.

The ideal $N_q(O_t)$ is generated by the matrix entries of the minimal polynomial of $\mathcal{Q}$ as an operator on $\mathbb{C}^7 \otimes M$, and the kernel of a central character of $\mathbb{C}_q[G]$ determined by the highest weight of M. It is expressed through q-traces of the powers of $\mathcal{Q}$. To study the minimal polynomial of $\mathcal{Q}$, we analyse the module structure of $\mathbb{C}^7 \otimes M$. Namely, we prove that $\mathbb{C}^7 \otimes M$ splits into a direct sum of submodules of highest weight. That guarantees semisimplicity of the matrix $\mathcal{Q}$ and helps to write down its minimal polynomial explicitly, which is crucial for our approach.

Our analysis of $\mathbb{C}^7 \otimes M$ consists of two parts. First we construct singular vectors in $\mathbb{C}^7 \otimes M$. For given $\mathfrak{k}$, the subset of points whose centralizer is $\mathfrak{k}$ forms an algebraic subset in T. That subset determines a subset of $\mathfrak{k}$-admissible weights parameterizing the module M of type $\mathfrak{k}$. We construct singular vectors in $\mathbb{C}^7 \otimes M$ as regular functions of weight that never turn zero. We do it in several steps. First we process the Verma module $M = \hat{M}$ and then get singular vectors in $\mathbb{C}^7 \otimes M$ for all other M. We do it by projection from $\mathbb{C}^7 \otimes \hat{M}$ with a subsequent regularization, because some singular vectors acquire scalar factors to be canceled.

The second part of our analysis identifies the conditions when the sum of submodules in $\mathbb{C}^7 \otimes M$ generated by the singular vectors exhausts all of $\mathbb{C}^7 \otimes M$ and when the sum is direct. This study makes use of ideas close to [19] and is based on the concept of extremal twist operator and its determinant.

The setup of the paper is as follows. In Section 2 we present a classification of semisimple conjugacy classes of G_2. It is followed by the basic information about the quantum group $U_q(\mathfrak{g})$ and its representation on $\mathbb{C}^7$ in Section 3. The quantization theorem is stated in Section 4, and is proved modulo direct sum decomposition of $\mathbb{C}^7 \otimes M$, which is completed in the subsequent sections. In Section 5, we establish some properties of generalized parabolic modules. The last four sections are more technical. In Section 6, we do regularization of singular vectors in $\mathbb{C}^7 \otimes M$ for a general Verma module M and decompose it into a direct sum of Verma modules. The subsequent sections are devoted to a similar study for all other types of M. Some useful formulas including the entries of a matrix participating in the Shapovalov inverse form can be found in Appendix.

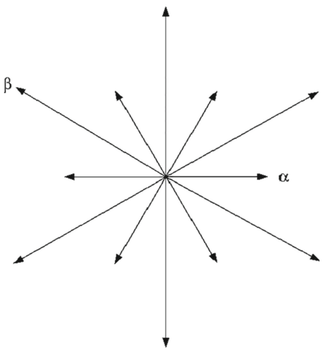
Throughout the paper we adopt the following general convention:

- For better readability of formulas, we denote a scalar inverse by the bar, e.g. $\bar{q} = q^{-1}$.
- The notation $a \simeq b$ means that a is proportional to b with a non-zero scalar factor. If the coefficient is a scalar function, we thereby assume that it never turns zero.

  The symbol $\simeq$ also stands for isomorphism, which is always clear from the context and causes no confusion.
- Divisibility by a regular scalar function ϕ is denoted by $\phi \sqsubset$.
- We use the notation $[z]_q = \dfrac{q^z - q^{-z}}{q - q^{-1}}$ whenever q^z makes sense.
- We assume that $q \in \mathbb{C}$ is not a root of unity.

2 Semisimple Conjugacy Classes of G_2

In this section we describe semisimple conjugacy classes of the complex algebraic group $G = G_2$. Let $\mathfrak{g}$ denote the Lie algebra of G with a fixed Cartan subalgebra $\mathfrak{h}$. Its root system R is displayed on the figure below.



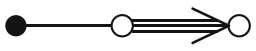
The subset of positive roots $\mathrm{R}^+ \subset \mathrm{R}$ contains the basis of simple roots $\Pi = \{\alpha, \beta\}$. We fix an inner product on $\mathfrak{h}^*$ so as

$$(\alpha, \alpha) = 2, \quad (\alpha, \beta) = -3, \quad (\beta, \beta) = 6.$$

The half-sum of positive roots $5\alpha + 3\beta$ is denoted by ρ.

To every $\lambda \in \mathfrak{h}^*$ we assign its image h_λ under the isomorphism $\mathfrak{h}^* \simeq \mathfrak{h}$ via the canonical form: $\mu(h_\lambda) = (\lambda, \mu)$ for all $\mu \in \mathfrak{h}^*$.

The group G_2 has a faithful representation in $\mathbb{C}^7$. The corresponding representation of the quantum group is given in Section 3.1.

The affine Dynkin diagram  of $\mathfrak{g}$ suggests the following stabilizers $\mathfrak{k} \subset \mathfrak{g}$ of semisimple conjugacy classes labelled by the their root bases $\Pi_\mathfrak{k} \subset \mathrm{R}^+_\mathfrak{g}$:

$\mathfrak{k}$		$\Pi_\mathfrak{k}$	
$\mathfrak{h}$		$\varnothing$	
$\mathfrak{k}_s$	$\{\alpha\}$,	$\{\alpha+\beta\}$,	$\{2\alpha+\beta\}$
$\mathfrak{k}_l$	$\{\beta\}$,	$\{3\alpha+\beta\}$,	$\{3\alpha+2\beta\}$
$\mathfrak{k}_{s,l}$	$\{\alpha, 3\alpha+2\beta\}$,	$\{\alpha+\beta, 3\alpha+\beta\}$,	$\{2\alpha+\beta, \beta\}$
$\mathfrak{k}_{l,l}$		$\{\beta, 3\alpha+\beta\}$	

The subscripts indicate the lengths of roots. There are three Levi types with $\#\Pi_\mathfrak{k} \leqslant 1$ and two pseudo-Levi types with $\#\Pi_\mathfrak{k} = 2$.

Different although isomorphic $\mathfrak{k}$ give rise to the same conjugacy class G/K, where the subgroup K with the Lie algebra $\mathfrak{k}$ is the centralizer of the initial point. Still we make this distinction because we associate with them different representations of quantized G/K.

Let T denote the maximal torus of G corresponding to $\mathfrak{h}$ and fix $t \in T$ such that $\mathfrak{k}$ is the centralizer of t. We parameterize T with a pair of non-zero complex coordinates $x, y \in \mathbb{C}^\star$. In the matrix realization that gives

$$t = \mathrm{diag}(xy, x, y, 1, \bar{y}, \bar{x}, \bar{y}\bar{x}) \in \mathrm{End}(\mathbb{C}^7). \tag{2.1}$$

Regarding the roots as characters on T, we have $\alpha(t) = y$, $\beta(t) = xy^{-1}$.

Define $T^\mathfrak{k} \subset T$ as the subset of points whose centralizer Lie algebra is $\mathfrak{k}$. We will also use the notation $T^{\Pi_\mathfrak{k}} = T^\mathfrak{k}$. We select the subset $T^\mathfrak{k}_{reg} \subset T^\mathfrak{k}$ of regular points, whose minimal polynomial in the representation on $\mathbb{C}^7$ has maximal degree. The complementary subset in $T^\mathfrak{k}$ is denoted by $T^\mathfrak{k}_{bord}$ and called borderline, see [1]. Such points are present only for $\mathfrak{k} = \mathfrak{h}$ and $\mathfrak{k} = \mathfrak{k}_l$. They are a sort of "transitional" from Levi to pseudo-Levi type, hence the name.

The set $T^\varnothing = T^\mathfrak{h}$ is determined by the conditions $x \neq 1$, $y \neq 1$, $x \neq y$, $xy^2 \neq 1$, $x^2y \neq 1$, $xy \neq 1$. Clearly, it is enough to use the first three diagonal matrix entries for description of t, so one can write:

$$T^\varnothing_{reg} : (xy, x, y),\ x \neq y,\ x^i y^j \neq 1,\ i, j = 0, 1, 2, \quad i + j > 0; \tag{2.2}$$

$$T^\varnothing_{bord} : (-x, -1, x), \quad (-x, x, -1), \quad (-1, x, -\bar{x}), \quad x^4 \neq 1; \tag{2.3}$$

$$T^\alpha : (x, x, 1),\ T^{\alpha+\beta} : (x, 1, x),\ T^{2\alpha+\beta} : (1, x, x^{-1}),\ x^2 \neq 1; \tag{2.4}$$

$$T^\beta_{reg} : (x^2, x, x),\ T^{3\alpha+\beta}_{reg} : (\bar{x}, \bar{x}^2, x),\ T^{3\alpha+2\beta}_{reg} : (\bar{x}, x, \bar{x}^2),\ x^3, x^4 \neq 1; \tag{2.5}$$

$$T^\beta_{bord} : (-1, x, x),\ T^{3\alpha+\beta}_{bord} : (\bar{x}, -1, x),\ T^{3\alpha+2\beta}_{bord} : (\bar{x}, x, -1),\ x = e^{\pm\frac{\pi i}{2}}; \tag{2.6}$$

$$T^{\alpha,3\alpha+2\beta} : (-1, -1, 1),\ T^{\alpha+\beta,3\alpha+\beta} : (-1, 1, -1),\ T^{2\alpha+\beta,\beta} : (1, -1, -1); \tag{2.7}$$

$$T^{\beta,3\alpha+\beta} : (\bar{x}, x, x),\ x = e^{\pm\frac{2\pi i}{3}}. \tag{2.8}$$

We have listed all possible $\mathfrak{k} \subset \mathfrak{g}$, so that the sets

$$T^s = T^\alpha \cup T^{\alpha+\beta} \cup T^{2\alpha+\beta}, \quad T^l = T^\beta \cup T^{3\alpha+\beta} \cup T^{3\alpha+2\beta},$$

$$T^{s,l} = T^{\alpha,3\alpha+2\beta} \cup T^{\alpha+\beta,3\alpha+\beta} \cup T^{2\alpha+\beta,\beta}, \quad T^{l,l} = T^{\beta,3\alpha+\beta},$$

along with $T^{\varnothing}$ and the group identity exhaust all of T. They consist of points whose conjugacy classes are isomorphic as homogeneous spaces.

Denote by $\tilde{G}$ the group $SL(7)$ and by $\tilde{W}$ its Weyl group. The Weyl group of G is denoted by W. It is an elementary fact that the intersection of a $\tilde{G}$-conjugacy class with G consists of a finite number of G-classes.

Proposition 2.1 *The conjugacy class of each semisimple element $t \in G$ is the intersection of its $\tilde{G}$-conjugacy class with G.*

Proof A semisimple conjugacy class of $\tilde{G}$ is determined by the set of eigenvalues and their multiplicities. It is sufficient to check that $\tilde{W}t \cap T = Wt$ for each $t \in T$.

Fix $t = (xy, x, y) \in T^{\varnothing}$ such that $y \neq -1$. One can check that the multiplicity of the eigenvalue y is 1. Present $\tilde{W}t \cap T$ as a union $Y \cup \bar{Y}$ of subsets whose elements have either y or $\bar{y}$ among their first three coordinates. This union is disjoint for $t \in T^{\varnothing}$.

One can check that $\#|Y| = \#|\bar{Y}| = 6$. Then $Y \subset Wt$ since

$$(xy, x, y) \overset{\sigma_\beta}{\sim} (xy, y, x) \overset{\sigma_\alpha}{\sim} (y, xy, \bar{x}) \overset{\sigma_\beta}{\sim} (y, \bar{x}, xy) \overset{\sigma_\alpha}{\sim} (\bar{x}, y, \bar{x}\bar{y}) \overset{\sigma_\beta}{\sim} (\bar{x}, \bar{x}\bar{y}, y).$$

The set $\bar{Y}$ is obtained from Y by inverting the coordinates. One has $\bar{Y} \subset Wt$ as $Y \ni (xy, x, y) \overset{\sigma_\alpha}{\sim} (x, xy, \bar{y}) \in \bar{Y}$.

Furthermore,

$$T^s \ni (x, 1, x) \overset{\sigma_{2\alpha+\beta}}{\sim} (\bar{x}, \bar{x}, 1) \overset{\sigma_\beta}{\sim} (\bar{x}, 1, \bar{x}) \overset{\sigma_{2\alpha+\beta}}{\sim} (x, x, 1) \overset{\sigma_{\alpha+\beta}}{\sim} (1, \bar{x}, x) \overset{\sigma_\beta}{\sim} (1, x, \bar{x}), \quad x^2 \neq 1,$$

$$T^l \ni (\bar{x}, x, \bar{x}^2) \overset{\sigma_\beta}{\sim} (\bar{x}, \bar{x}^2, x) \overset{\sigma_\alpha}{\sim} (\bar{x}^2, \bar{x}, \bar{x}) \overset{\sigma_{\beta+2\alpha}}{\sim} (x^2, x, x) \overset{\sigma_\alpha}{\sim} (x, x^2, \bar{x}) \overset{\sigma_\beta}{\sim} (x, \bar{x}, x^2), \quad x^2, x^3 \neq 1,$$

$$T^{l,s} \ni (1, -1, -1) \overset{\sigma_\alpha}{\sim} (-1, 1, -1) \overset{\sigma_\beta}{\sim} (-1, -1, 1),$$
$$T^{l,l} \ni (e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}) \overset{\sigma_\alpha}{\sim} (e^{-\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}}).$$

This proves that $\tilde{W}t = Wt$ for each $t \in T$. □

A semisimple $\tilde{G}$-class is determined by the set of eigenvalues and their multiplicities. The eigenvalues are fixed by the minimal polynomial while the multiplicities by the character of the subalgebra of invariants under conjugation. The subalgebra of invariants when restricted to maximal torus $\tilde{T} \subset \tilde{G}$ is generated by the functions $t \mapsto \mathrm{Tr}(t^m)$, $m = 1, \ldots, 7$, and the character is evaluation at t. All possible minimal polynomials of $t \in T$ are listed here:

$$(t - xy)(t - x)(t - y)(t - 1)(t - \bar{y})(t - \bar{x})(t - \bar{y}\bar{x}), \quad t \in T^{\varnothing}_{reg}, \quad (x, y) \in \mathbb{C}^2_{reg},$$
$$(t^2 - x^2)(t^2 - 1)(t^2 - \bar{x}^2), \quad t \in T^{\varnothing}_{bord}, \quad x^4 \neq 1, \quad \mathrm{mult}(-1) = 2,$$
$$(t - x)(t - 1)(t - \bar{x}), \quad t \in T^s, \quad x^2 \neq 1, \quad \mathrm{mult}(x^{\pm 1}) = 2,$$
$$(t - x^2)(t - x)(t - 1)(t - \bar{x})(t - \bar{x}^2), \quad t \in T^l_{reg}, \quad x^3 \neq 1 \neq x^4, \quad \mathrm{mult}(x^{\pm 1}) = 2,$$
$$(t^4 - 1), \quad t \in T^l_{bord}, \quad \mathrm{mult}(-1) = \mathrm{mult}(\pm i) = 2,$$
$$(t^2 - 1), \quad t \in T^{s,l}, \quad \mathrm{mult}(-1) = 4,$$
$$(t^3 - 1), \quad t \in T^{l,l}, \quad \mathrm{mult}(e^{\pm \frac{2\pi i}{3}}) = 3.$$

Remark that regular points, contrary to borderline, separate irreducible $\mathfrak{k}$-submodules in $\mathbb{C}^7$. The two bottom lines correspond to the two pseudo-Levi classes.

Corollary 2.2 *The ideal $I = N(O_t)$ of a semisimple conjugacy class O_t in $\mathbb{C}[G]$ is generated by the entries of the minimal polynomial over the maximal ideal of the subalgebra of invariants.*

Proof Fix a semisimple point $t \in G$ and consider its conjugacy classes $\tilde{O}_t \subset \tilde{G}$ and $O_t \subset G$. Let F_1 and F_2 be G-submodules in $\mathbb{C}\left[\mathrm{End}(\mathbb{C}^7)\right]$ generating the ideals $N(\tilde{O}_t)$ and $N(G)$, respectively. Put $f_i\colon \mathrm{End}(\mathbb{C}^7) \to F_i^*$ to be the corresponding G-invariant maps and set $f = f_1 \oplus f_2$. By Proposition 2.1, it suffices to prove that $\ker df(t) = \ker df_1(t) \cap \ker df_2(t)$ has the same dimension as O_t, cf. [17], Prop. 2.1.

Identify the tangent space at t with $\tilde{\mathfrak{g}}$ via the left translation by t, then $\ker df_2(t) = \mathfrak{g}$. For the general linear group, $\ker df_1(t) = \tilde{\mathfrak{m}}_t = \tilde{\mathfrak{g}} \ominus \tilde{\mathfrak{k}}$, where $\tilde{\mathfrak{m}}_t$ is the sum of Ad_t-eigenspaces in $\tilde{\mathfrak{g}}$ corresponding to eigenvalues disctinct from 1 and $\tilde{\mathfrak{k}}$ is the centralizer of t. Then $\ker df(t) = \ker df_1(t) \cap \ker df_2(t) = \tilde{\mathfrak{m}}_t \cap \mathfrak{g} = \mathfrak{g} \ominus \mathfrak{k}$ since $t \in G$ and it is semi-simple. □

3 Quantized Universal Enveloping Algebra

Throughout the paper we assume that $q \in \mathbb{C}$ is a non-zero complex number that is not a root of unity. Denote by $U_q(\mathfrak{g}_\pm)$ the $\mathbb{C}$-algebra generated by $e_{\pm\alpha}, e_{\pm\beta}$ subject to the q-Serre relations

$$e_{\pm\alpha}^4 e_{\pm\beta} - [4]_q e_{\pm\alpha}^3 e_{\pm\beta} e_{\pm\alpha} + [3]_q (q^2 + q^{-2}) e_{\pm\alpha}^2 e_{\pm\beta} e_{\pm\alpha}^2 - [4]_q e_{\pm\alpha} e_{\pm\beta} e_{\pm\alpha}^3 + e_{\pm\beta} e_{\pm\alpha}^4 = 0,$$
$$e_{\pm\beta}^2 e_{\pm\alpha} - (q^3 + q^{-3}) e_{\pm\beta} e_{\pm\alpha} e_{\pm\beta} + e_{\pm\alpha} e_{\pm\beta}^2 = 0.$$

Denote by $U_q(\mathfrak{h})$ the commutative $\mathbb{C}$-algebra generated by $\{q^{\pm h_\alpha}\}_{\alpha \in \Pi}$, with $q^{h_\alpha} q^{-h_\alpha} = 1$. The quantum group $U_q(\mathfrak{g})$ is a $\mathbb{C}$-algebra generated by $U_q(\mathfrak{g}_\pm)$ and $U_q(\mathfrak{h})$ subject to the relations [7]:

$$q^{h_\alpha} e_{\pm\alpha} q^{-h_\alpha} = q^{\pm 2} e_{\pm\alpha}, \quad q^{h_\alpha} e_{\pm\beta} q^{-h_\alpha} = q^{\mp 3} e_{\pm\beta},$$
$$q^{h_\beta} e_{\pm\alpha} q^{-h_\beta} = q^{\mp 3} e_{\pm\alpha}, \quad q^{h_\beta} e_{\pm\beta} q^{-h_\beta} = q^{\pm 6} e_{\pm\beta},$$
$$[e_\alpha, e_{-\alpha}] = [h_\alpha]_q, \quad [e_\beta, e_{-\beta}] = \frac{1}{[3]_q}[h_\beta]_q,$$
$$[e_{\pm\alpha}, e_{\mp\beta}] = 0.$$

Remark that the vector space $\mathfrak{h}$ is not contained in $U_q(\mathfrak{g})$, still it is convenient for calculations to keep reference to $\mathfrak{h}$.

The comultiplication on the generators is defined as follows:

$$\Delta(e_\mu) = e_\mu \otimes 1 + q^{h_\mu} \otimes e_\mu, \quad \Delta(e_{-\mu}) = 1 \otimes e_{-\mu} + e_{-\mu} \otimes q^{-h_\mu},$$
$$\Delta(q^{\pm h_\mu}) = q^{\pm h_\mu} \otimes q^{\pm h_\mu},$$

for all $\mu \in \Pi$. It is opposite to that in [4].

We will use the notation $f_\alpha = e_{-\alpha}$, $f_\beta = e_{-\beta}$.

The subalgebras in $U_q(\mathfrak{g})$ generated by $U_q(\mathfrak{g}_\pm)$ over $U_q(\mathfrak{h})$ are quantized universal enveloping algebras of the Borel subalgebras $\mathfrak{b}_\pm = \mathfrak{h} + \mathfrak{g}_\pm \subset \mathfrak{g}$ denoted further by $U_q(\mathfrak{b}_\pm)$.

The Chevalley generators can be supplemented with root vectors for compound $\mu \in \mathrm{R}^+$. They participate in construction of a Poincaré-Birkhoff-Witt (PBW) basis in $U_q(\mathfrak{g})$ and universal $\mathcal{R}$-matrix, [4].

The universal R-matrix is an element of a certain extension of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. Let $\{\varepsilon_i\}_{i=1}^2$ be an orthogonal basis in $\mathfrak{h}^*$. The exact expression for $\mathcal{R}$ (up to the flip of tensor legs) is extracted from [4], Theorem 8.3.9:

$$\mathcal{R} = q^{\sum_{i=1}^2 h_{\varepsilon_i} \otimes h_{\varepsilon_i}} \prod_{\mu \in \mathrm{R}^+} \exp_{q_\mu}(1 - q_\mu^{-2})(f_\mu \otimes e_\mu) \in U_q(\mathfrak{b}_-) \hat{\otimes} U_q(\mathfrak{b}_+), \tag{3.9}$$

where $\exp_q(x)=\sum_{k=0}^{\infty} q^{\frac{1}{2}k(k+1)}\frac{x^k}{[k]_q}$, $q_\mu=q^{\frac{(\mu,\mu)}{2}}$, and the product is ordered in a certain way. Its reduction to the minimal representation can be found in [21, 22].

We will also work with the $\mathbb{C}[[\hbar]]$-extension of $U_q(\mathfrak{g})$ completed in the $\hbar$-adic topology, which we denote by $U_\hbar(\mathfrak{g})$. Note that the Cartan subalgebra in $U_\hbar(\mathfrak{g})$ is isomorphic to the polynomial algebra on $\mathfrak{h}$ while $U_q(\mathfrak{h})$ is that on T.

3.1 Minimal Representation of $U_q(\mathfrak{g})$

In this section we describe a representation of $U_q(\mathfrak{g})$ on the vector space $\mathbb{C}^7$. It is a deformation of the classical representation of $\mathfrak{g}$ restricted from $\mathfrak{so}(7)$. Our realization is close to [14].

Let I denote the set of integers from $1,\dots,7$. Fix a basis $\{w_i\}_{i\in I}\in\mathbb{C}^7=V$ and let $e_{ij}\in\mathrm{End}(V)$ denote the standard matrix units, $e_{ij}w_k=\delta_{jk}w_i$, $i,j,k\in I$. One can check that the assignment

$$
\begin{aligned}
q^{h_\alpha} &\mapsto qe_{11}+q^{-1}e_{22}+q^2e_{33}+e_{44}+q^{-2}e_{55}+qe_{66}+q^{-1}e_{77},\\
q^{h_\beta} &\mapsto e_{11}+q^3e_{22}+q^{-3}e_{33}+e_{44}+q^3e_{55}+q^{-3}e_{66}+e_{77},
\end{aligned}
$$

$$
\begin{aligned}
e_\alpha &\mapsto e_{12}+e_{34}+e_{45}+e_{67}, & f_\alpha &\mapsto e_{21}+[2]_qe_{43}+[2]_qe_{54}+e_{76},\\
e_\beta &\mapsto e_{23}+e_{56}, & f_\beta &\mapsto e_{32}+e_{65}.
\end{aligned}
$$

is compatible with the defining relations and extends to a homomorphism $U_q(\mathfrak{g})\to\mathrm{End}(V)$.

Up to scalar multiplies, the action of $U_q(\mathfrak{g}_-)$ can be depicted by the graph

$$
\mathfrak{g}_-
$$

$$
\underset{w_1}{\circ}\xrightarrow{f_\alpha}\underset{w_2}{\circ}\xrightarrow{f_\beta}\underset{w_3}{\circ}\xrightarrow{f_\alpha}\underset{w_4}{\circ}\xrightarrow{f_\alpha}\underset{w_5}{\circ}\xrightarrow{f_\beta}\underset{w_6}{\circ}\xrightarrow{f_\alpha}\underset{w_7}{\circ}
$$

The representation of $U_q(\mathfrak{g}_+)$ is obtained by reversing the arrows. Let $\nu_i\in\mathfrak{h}^*$ denote the weight of w_i, then

$$
\nu_1=2\alpha+\beta,\quad \nu_2=\alpha+\beta,\quad \nu_3=\alpha,\quad \nu_4=0,\quad \nu_5=-\alpha,\quad \nu_6=-\alpha-\beta,\quad \nu_7=-2\alpha-\beta.
$$

For all $\nu\in\mathbb{Z}\Pi$ we denote by $P(\nu)$ the set of pairs $(i,j)\in I\times I$ such that $\nu_i-\nu_j=\nu$. For each pair with $i<j$ there is a unique monomial ψ in f_α, f_β such that $w_j=\psi w_i$. Let $\psi^{ij}\in U_q(\mathfrak{g}_-)$ be the monomial obtained from it by reversing the order of factors.

4 Quantum Conjugacy Classes

In this section we describe quantum semisimple conjugacy classes using ideas of [16] and [2]. The construction is based on certain facts from representation theory to be established in the subsequent sections.

We regard roots as multiplicative characters of T and elements of T as spectral points of $U_q(\mathfrak{h})$ via the correspondence $t\colon q^{h_\alpha}\mapsto\alpha(t)$, for all $t\in T$. Fix $t\in T$ and its stabilizer subalgebra $\mathfrak{k}$. Put $t_q=tq^{2\rho_\mathfrak{k}-2\rho}\in T$ and choose the weight λ from the condition $q^{2\lambda}=t_q$ regarded as an equality in T upon the identification $\mathfrak{h}\simeq\mathfrak{h}^*$ via the inner product. Consider a 1-dimensional representation $\mathbb{C}_\lambda$ of $U_q(\mathfrak{b}_+)$ extending $t_q\colon U_q(\mathfrak{h})\to\mathbb{C}$: it is implemented by the assignment $q^{\pm h_\alpha}\mapsto q^{\pm(\lambda,\alpha)}$, $e_\alpha\mapsto 0$. Let $M_\lambda=U_q(\mathfrak{g})\otimes_{U_q(\mathfrak{g}_+)}\mathbb{C}_\lambda$ denote the Verma $U_q(\mathfrak{g})$-module of highest weight λ with the canonical generator $v_\lambda\in M_\lambda$.

Due to the special choice of λ, it satisfies the conditions $q^{2(\lambda+\rho,\alpha)} = q^{(\alpha,\alpha)}$ for all $\alpha \in \Pi_{\mathfrak{k}}$, and we denote by $\Theta^{\mathfrak{k}}$ the set of all such weights. For each $\alpha \in \Pi_{\mathfrak{k}}$ there exists a singular vector $v_{\lambda-\alpha} \in M_\lambda$ of weight $\lambda - \alpha$ annihilated by e_β for all $\beta \in \Pi$. Up to a scalar multiplier, they can be written explicitly as $\hat{f}_{ij} v_\lambda$ with $(i, j) \in P(\alpha)$, where $\hat{f}_{ij}$ are matrix elements of the (reduced) Shapovalov inverse [18]. For our case, they are defined in Eq. 5.12. The vectors $v_{\lambda-\alpha}$ generate submodules $M_{\lambda-\alpha} \subset M_\lambda$. Let $M_\lambda^{\mathfrak{k}}$ denote the quotient module $M_\lambda / \sum_{\alpha\in\Pi_{\mathfrak{k}}} M_{\lambda-\alpha}$. On transition to the $\mathbb{C}[[\hbar]]$-extension via $q = e^\hbar$, $M_\lambda^{\mathfrak{k}}$ becomes a free module over $\mathbb{C}[[\hbar]]$, by Lemma 5.2.

Given a $\mathbb{C}$-algebra $\mathcal{A}$ we call a $\mathbb{C}[[\hbar]]$-algebra $\mathcal{A}_\hbar$ quantization of $\mathcal{A}$ if it is a free $\mathbb{C}[[\hbar]]$-module and $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$ is isomorphic to $\mathcal{A}$ as a $\mathbb{C}$-algebra. The quantization is called equivariant if $\mathcal{A}_\hbar$ is a $U_\hbar(\mathfrak{g})$-module algebra such that the $U_\hbar(\mathfrak{g})$-action is a deformation of a $U(\mathfrak{g})$-action on $\mathcal{A}$. If a $U_\hbar(\mathfrak{g})$-algebra $\mathcal{A}_\hbar$ is a $\mathbb{C}[[\hbar]]$-extension of a $U_q(\mathfrak{g})$-algebra $\mathcal{A}_q$, then $\mathcal{A}_q$ is also called (equivariant) quantization of $\mathcal{A}$. This should not be confusing as the ring of scalars is always explicitly stated.

The key role in our approach to quantization of conjugacy classes belongs to an element $\mathcal{Q} = (\pi \otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R}) \in \mathrm{End}(V) \otimes U_q(\mathfrak{g})$, which is a matrix with entries in the quantum group. It commutes with the coproduct of all elements from $U_q(\mathfrak{g})$ and plays a key role in the theory. Its entries generate an ad-invariant subalgebra $\mathbb{C}_q[G] \subset U_q(\mathfrak{g})$, which is a quantization of the $U(\mathfrak{g})$-algebra $\mathbb{C}[G]$ equipped with the conjugation action. So $\mathcal{Q}$ is the matrix of "quantum coordinate" functions on G. It satisfies the so called "reflection equation" [6, 13] rather than the RTT-relations of the Hopf dual to $U_q(\mathfrak{g})$, [10].

The operator $\mathcal{Q}$ is scalar on every submodule of highest weight in $V \otimes M_\lambda^{\mathfrak{k}}$ as well as a quotient module. In the classical limit, the module V is completely reducible over $\mathfrak{k}$. Let $I^{\mathfrak{k}} \subset I$ denote the subset of indices of $\mathfrak{k}$-highest vectors w_i. Then the eigenvalues of $\mathcal{Q}$ on $V \otimes M_\lambda^{\mathfrak{k}}$ are $x_j = q^{2(\lambda+\rho,\nu_j)+(\nu_j,\nu_j)-(\nu_1,\nu_1)-2(\rho,\nu_1)}$, $j \in I^{\mathfrak{k}}$, and they obviously deform the eigenvalues $\nu_j(t)$ of $t \in O_t$. A proof for the case $\mathfrak{k} = \mathfrak{h}$ can be found in [16] (valid for all simple groups), while for general $\mathfrak{k}$ it readily follows from Proposition 5.4. We prove in the subsequent sections that $\mathcal{Q}$ is semisimple on $V \otimes M_\lambda^{\mathfrak{k}}$, so its minimal polynomial is

$$\prod_{j\in I^{\mathfrak{k}}}{}^{\prime} (\mathcal{Q} - x_j) = 0, \tag{4.10}$$

where the prime means that only distinct eigenvalues count (coincidences occur only for borderline t).

When t varies in a W-orbit, the highest weight of $M_\lambda^{\mathfrak{k}}$ varies in an orbit under the shifted action. It follows from Proposition 5.4 that the eigenvalues of $\mathcal{Q}$ are determined by weights $\{\nu_j + \lambda\}_{j\in I^{\mathfrak{k}}}$, where $\{\nu_j\}_{j\in I^{\mathfrak{k}}}$ are highest weights of $\mathfrak{k}$-submodules in V. Since the ordinary W-action relates the sets $\{\nu_j\}_{j\in I^{\mathfrak{k}}}$, the shifted action relates the sets $\{\nu_j + \lambda\}_{j\in I^{\mathfrak{k}}}$. On the other hand, $\mathcal{Q}$ can be written as $\Delta(z)(z^{-1} \otimes z^{-1})$ for a central element $z \in U_q(\mathfrak{g})$, [8]. Therefore the set of eigenvalues $\{x_j\}_{j\in I^{\mathfrak{k}}}$ and the minimal polynomial (4.10) depend only on the class of the point t.

Put $\mathrm{Tr}_q(X) = \mathrm{Tr}\left(\pi(q^{2h_\rho})X\right)$ for a 7×7 matrix X with arbitrary entries. Then the elements $\tau^m = \mathrm{Tr}_q(\mathcal{Q}^m)$, $m \in I$, generate the center of $\mathbb{C}_q[G]$. Note that τ^m are not independent, as the rank of G is 2.

Let χ_λ denote the central character of $\mathbb{C}_q[G]$ returning $\tau^m v_\lambda = \chi_\lambda(\tau^m) v_\lambda$. It is expressed through the eigenvalues $\{x_i\}_{i=1}^7$ of $\mathcal{Q}$ on the tensor product of V with the Verma module of highest weight λ. The operator $\mathcal{Q}$ is a scalar multiple and returns x_i on the submodule in $V \otimes M_\lambda$ of the highest weight $\nu_i + \lambda$, $i = 1, \ldots, 7$. Explicitly [16],

$$\chi_\lambda(\tau^m) = \sum_{i=1}^{7} x_i^m \prod_{\alpha \in \mathrm{R}_+} \frac{q^{(\lambda+\rho+\nu_i,\alpha)} - q^{-(\lambda+\rho+\nu_i,\alpha)}}{q^{(\lambda+\rho,\alpha)} - q^{-(\lambda+\rho,\alpha)}}. \tag{4.11}$$

It is invariant under the shifted action of the Weyl group W on $\mathfrak{h}^*$.

Clearly $\ker \chi_\lambda$ belongs to the annihilator of $M_\lambda^{\mathfrak{k}}$. In the subsequent sections we prove that the minimal polynomial of $\mathcal{Q}$ regarded as an operator on $V \otimes M_\lambda^{\mathfrak{k}}$ is a deformation of the minimal polynomial of O_t, so the relations (4.10) and (4.11) deform the defining relations of O_t. Based on this fact, we formulate the main result of this work.

Theorem 4.1 *The image of* $\mathbb{C}_q[G]$ *in* $\mathrm{End}(M_\lambda^{\mathfrak{k}})$ *is an equivariant quantization of the algebra* $\mathbb{C}[O_t]$. *It is the quotient of* $\mathbb{C}_q[G]$ *by the ideal generated by the entries of the minimal polynomial in* $\mathcal{Q}$, *over the kernel of* χ_λ. *The algebra* $\mathbb{C}_q[O_t]$ *depends only on the* W*-orbit of the point* $t \in T$.

Proof We extend the $U_q(\mathfrak{g})$-module algebras $\mathbb{C}_q[G]$ and $\mathrm{End}(M_\lambda^{\mathfrak{k}})$ over $\mathbb{C}[[\hbar]]$ to make them $U_\hbar(\mathfrak{g})$-module algebras, respectively, $\mathbb{C}_\hbar[G]$ and $\mathcal{T}$.

Every isotypic component of $\mathbb{C}_\hbar[G]$ is finitely generated over its centre (a q-version of the Richardson theorem [15]). Then the quotient $\mathcal{S}_\hbar$ of $\mathbb{C}_\hbar[G]$ by the ideal generated by $\ker \chi_\lambda$ is a direct sum of finite isotypic components, and its image in $\mathcal{T}$ is $\mathbb{C}[[\hbar]]$-free since multiplication by $\hbar$ is injective on $\mathcal{T}$. Then the kernel $\mathcal{J}_\hbar \subset \mathcal{S}_\hbar$ is a direct summand, and one has an $\mathfrak{g}$-invariant embedding $\mathcal{J}_0 \subset \mathcal{S}_0$ of the zero fibers modulo $\hbar$.

Let $\mathcal{J}'_\hbar \subset \mathcal{J}_\hbar$ denote the ideal generated by the relations (4.10) and (4.11). In the classical limit modulo $\hbar$ we have a map $\mathcal{J}'_0 \to \mathcal{J}_0$. Its image coincides with the image of $N(O_t)$ in $\mathcal{S}_0$, because the relations (4.10) and (4.11) go over to the defining relations of $N(O_t)$. Since the image of $N(O_t)$ in $\mathcal{S}_0$ is a maximal $\mathfrak{g}$-invariant ideal in $\mathcal{S}_0$, the map $\mathcal{J}'_0 \to \mathcal{J}_0$ is surjective. By the Nakayama lemma the embedding $\mathcal{J}'_\hbar \to \mathcal{J}_\hbar$ is an isomorphism, and the annihilator of $M_\lambda^{\mathfrak{k}}$ in $\mathbb{C}_q[G]$ is generated by Eqs. 4.10 and 4.11.

The relations (4.10) and (4.11) are invariant under the shifted action of W on the weight λ. Therefore $\mathcal{J}_\hbar$ is W-invariant, and $\mathbb{C}_q[O_t]$ depends only on the class of t. □

The key fact underlying the above reasoning is semisimplicity of $\mathcal{Q}$. To complete the proof, it is sufficient to show that the sum of highest weight submodules exhausts all of $V \otimes M_\lambda^{\mathfrak{k}}$. We solve a stronger problem: we establish exact criteria when $V \otimes M_\lambda^{\mathfrak{k}}$ splits into a direct sum of highest weight modules. The rest of the paper is devoted to this analysis.

5 Generalized Parabolic Verma Modules

Fix $\lambda \in \mathfrak{h}^*$ and consider the Verma module M_λ of highest weight λ. Let ${}^*M_\lambda$ denote the opposite Verma module of lowest weight $-\lambda$. There is a unique, up to a scalar multiple, $U_q(\mathfrak{g})$-invariant form $M_\lambda \otimes {}^*M_\lambda \to \mathbb{C}$ (equivalent to the contravariant Shapovalov on M_λ), which is non-degenerate if and only if M_λ is irreducible. As that is the case for generic weight, [5], there is a unique lift $\hat{\mathcal{F}} \in U_q(\mathfrak{g}_+)\hat{\otimes} U_q(\mathfrak{b}_-)$ of the inverse form, where the Borel subalgebra is extended over the ring of fractions of $U_q(\mathfrak{h})$ and the tensor product is completed with series. The matrix $\hat{F} = (\pi \otimes \mathrm{id})(\hat{\mathcal{F}}) \in \mathrm{End}(V) \otimes U_q(\mathfrak{b}_-)$ is expressed through another matrix $F \in \mathrm{End}(V) \otimes U_q(\mathfrak{g}_-)$ whose entries f_{ij} are certain polynomial in the Chevalley generators. They are "extracted" from an R-matrix of $U_q(\mathfrak{g})$, and their explicit expressions are presented in Appendix A.2.

Put $\rho_i = (\rho, \nu_i)$, $\tilde{\rho}_i = \rho_i + \frac{1}{2}||\nu_i||^2$, $i \in I$, and define

$$\eta_{ij} = h_{\nu_i} - h_{\nu_j} + \rho_i - \rho_j - \frac{||\nu_i - \nu_j||^2}{2}, \quad A_i^j = \frac{q - q^{-1}}{1 - q^{2\eta_{ij}}}, \quad i < j.$$

For an ascending sequence of integers $m_1, \ldots, m_k, j$, put $f_{m_1,\ldots,m_k} = f_{m_1,m_2} \ldots f_{m_{k-1},m_k}$ and $A^j_{m_1,\ldots,m_k} = A^j_{m_1} \ldots A^j_{m_k}$. Then $\hat{F} = \sum_{i,j=1}^7 e_{ij} \otimes \hat{f}_{ij}$, with

$$\hat{f}_{ij} = \sum_{i<\vec{m}<j} f_{i,\vec{m},j} A^j_{i,\vec{m}} q^{\eta_{ij} - \tilde{\rho}_i + \tilde{\rho}_j}, \tag{5.12}$$

where the summation is taken over all sequences $\vec{m} = (m_1, \ldots, m_k)$ such that $i < m_1$, $m_k < j$, including $\vec{m} = \varnothing$. Finally, we set $\hat{f}_{ii} = 1$ for all i and $\hat{f}_{ij} = 0$ for $i > j$.

We regard $U_q(\mathfrak{h})$ as the algebra of trigonometric polynomials on $\mathfrak{h}^*$. The linear isomorphism $U_q(\mathfrak{b}_-) \simeq U_q(\mathfrak{g}_-) \otimes U_q(\mathfrak{h})$ identifies elements of $U_q(\mathfrak{b}_-)$ with functions $\mathfrak{h}^* \to U_q(\mathfrak{g}_-)$.

A singular vector $v_{\lambda-\alpha} \in M_\lambda$ of weight $\lambda - \alpha$ can be constructed as follows. It is known to be unique, up to a scalar factor. Therefore it is proportional to $\hat{f}_{ij} v_\lambda = \hat{f}_{ij}(\lambda) v_\lambda$ with $(i, j) \in P(\alpha)$, upon an appropriate regularization of $\hat{f}_{ij}(\lambda)$ if needed (the element $\hat{f}_{ij}$ can be also constructed using reduction algebras, [12]).

Lemma 5.1 *Suppose that $t \in T^{\mathfrak{k}}$ and $\lambda = \frac{1}{2} \ln t_q \in \mathfrak{h}^*$ is fixed as in* Section 4. *For all $\alpha \in \Pi_{\mathfrak{k}}$ choose a pair $(i, j) \in P(\alpha)$. Then $\hat{f}_{ij}(\lambda) \in U_q(\mathfrak{g}_-)$ is a deformation of a classical root vector, $f_\alpha \in \mathfrak{g}_-$.*

Proof The proof is based on the fact that in any presentation of α as a sum of positive roots the summands do not belong to $R^+_{\mathfrak{k}}$, see e.g. [2], Lemma 2.2. □

Corollary 5.2 *The $\mathbb{C}[[\hbar]]$-extension of $M^{\mathfrak{k}}_\lambda$ is $\mathbb{C}[[\hbar]]$-free.*

Proof The proof is similar to [17], Proposition 6.2. □

5.1 Standard Filtration of $V \otimes M_\lambda$

Define $\mathcal{V}_j \subset V \otimes M_\lambda$, $j \in I$, to be the submodule generated by $\{w_i \otimes v_\lambda\}_{i=1}^j$. They form an ascending filtration $\mathcal{V}_\bullet$ of $V \otimes M_\lambda$, which we call standard. Its graded module $\mathrm{gr}\, \mathcal{V}_\bullet$ is the direct sum $\oplus_{i=1}^j \mathrm{gr}\, \mathcal{V}_j$, where $\mathrm{gr}\, \mathcal{V}_j = \mathcal{V}_j / \mathcal{V}_{j-1}$ is isomorphic to $M_{\lambda+\nu_j}$ for all λ (the proof is similar to [3] for classical $U(\mathfrak{g})$). It is generated by the image w^j_λ of $w_j \otimes v_\lambda$ in $\mathrm{gr}\, \mathcal{V}_j$.

Proposition 5.3 *Suppose $(i, j) \in P(\beta)$, and ψ is a Chevalley monomial of weight $-\beta$. If $\psi \not\simeq \psi^{ij}$, then $w_i \otimes \psi v_\lambda \in \mathcal{V}_{j-1}$. Otherwise,*

$$w_i \otimes \psi^{ij} v_\lambda \simeq w_j \otimes v_\lambda \mod \mathcal{V}_{j-1}. \tag{5.13}$$

Proof The proof is similar to [2], Proposition 3.5. □

Assuming $\lambda \in \Theta^{\mathfrak{k}}$ denote by $\mathcal{V}^{\mathfrak{k}}_j$ the image of $\mathcal{V}_j$ under the projection $V\otimes M_\lambda \to V\otimes M^{\mathfrak{k}}_\lambda$. Clearly the sequence $\mathcal{V}^{\mathfrak{k}}_\bullet = (\mathcal{V}^{\mathfrak{k}}_j)$ forms an ascending filtration of $V\otimes M^{\mathfrak{k}}_\lambda$. Denote by $\bar I^{\mathfrak{k}}$ the complement of $I^{\mathfrak{k}}$ in I. Then $j \in \bar I^{\mathfrak{k}}$ if and only if there is $i < j$ such that $\nu_i - \nu_j \in \Pi_{\mathfrak{k}}$.

Proposition 5.4 *The graded module* $\mathrm{gr}\,\mathcal{V}^{\mathfrak{k}}_\bullet$ *of the filtration* $\mathcal{V}^{\mathfrak{k}}_\bullet$ *is isomorphic to* $\oplus_{j\in I^{\mathfrak{k}}}\mathrm{gr}\,\mathcal{V}^{\mathfrak{k}}_j$.

Proof Fix j and put and $\beta = \nu_1-\nu_j$. The module $\mathrm{gr}\mathcal{V}^{\mathfrak{k}}_j$ is a quotient of $\mathcal{V}_j$ by the submodule $\mathcal{V}_{j-1} + (V\otimes M)\cap \mathcal{V}_j$, where $M = \sum_{\alpha\in\Pi_{\mathfrak{k}}} M_{\lambda-\alpha}$. By Proposition 5.3, its subspace of weight $\nu_j+\lambda$ is isomorphic to the quotient of $w_1\otimes M_\lambda[\lambda-\beta]$ by $w_1\otimes \mathcal{N}_\beta v_\lambda + w_1\otimes M[\lambda-\beta]$, where $\mathcal{N}_\beta \subset U_q(\mathfrak{g}_-)$ is spanned by Chevalley monomials of weight $-\beta$ that are not proportional to ψ^{1j}. Then $M_\lambda[\lambda-\beta] = \mathcal{N}_\beta v_\lambda + M[\lambda-\beta]$ if and only if $\psi^{1j}v_\lambda \in M[\lambda-\beta]$ mod $\mathcal{N}_\beta v_\lambda$, which is equivalent to $j\in \bar I^{\mathfrak{k}}$. Otherwise $\mathrm{gr}\,\mathcal{V}^{\mathfrak{k}}_j[\lambda-\beta]$ is spanned by the image of $w_1\otimes\psi^{1j}v_\lambda \simeq w_j\otimes v_\lambda$, the generator of $\mathrm{gr}\,\mathcal{V}^{\mathfrak{k}}_j$. This proves that $\mathcal{V}^{\mathfrak{k}}_j = \mathcal{V}^{\mathfrak{k}}_{j-1}$ for each $j\in\bar I^{\mathfrak{k}}$ and $\mathcal{V}^{\mathfrak{k}}_j/\mathcal{V}^{\mathfrak{k}}_{j-1} \neq \{0\}$ for $j\in I^{\mathfrak{k}}$. □

Let $M_j \subset V\otimes M_\lambda$ denote the submodule of highest weight $\lambda+\nu_j$ and let u_j be its highest weight generator. Furthermore, consider $M^{\mathfrak{k}}_\lambda$ for $\lambda\in\Theta^{\mathfrak{k}}$ and let $\pi^{\mathfrak{k}}_\lambda$ denote the projection $V\otimes M_\lambda \to V\otimes M^{\mathfrak{k}}_\lambda$. Define $M^{\mathfrak{k}}_j = \pi^{\mathfrak{k}}_\lambda(M_j)$ and $\mathcal{W}^{\mathfrak{k}}_j = \sum_{i=1}^j M^{\mathfrak{k}}_i$. The ascending sequence $\mathcal{W}^{\mathfrak{k}}_\bullet = (\mathcal{W}^{\mathfrak{k}}_j)$, $j=1,\ldots,7$, of submodules is also invariant under the action of $\mathcal{Q}$, which is semisimple on $W^{\mathfrak{k}}_7$. Semi-simplicity of $\mathcal{Q}$ is important for our studies, so the question is when $V\otimes M^{\mathfrak{k}}_\lambda = \mathcal{W}^{\mathfrak{k}}_7$ or, more specifically, $V\otimes M^{\mathfrak{k}}_\lambda = \oplus_{j\in I^{\mathfrak{k}}} M^{\mathfrak{k}}_j$. We answer this question by comparing $\mathcal{W}^{\mathfrak{k}}_\bullet$ against $\mathcal{V}^{\mathfrak{k}}_\bullet$. First of all, observe that $\mathcal{W}^{\mathfrak{k}}_j \subset \mathcal{V}^{\mathfrak{k}}_j$, by Proposition 5.3.

Proposition 5.5 *Suppose that* $\mathfrak{k}$ *is Levi and fix* $j\in I^{\mathfrak{k}}$. *Then the following statements are equivalent: i)* $\mathcal{V}^{\mathfrak{k}}_j = \mathcal{W}^{\mathfrak{k}}_j$, *ii)* $\mathcal{V}^{\mathfrak{k}}_i = \mathcal{W}^{\mathfrak{k}}_i$ *for all* $i\leqslant j$, *iii) projection* $\wp^{\mathfrak{k}}_i\colon M^{\mathfrak{k}}_i \to \mathrm{gr}\,\mathcal{V}^{\mathfrak{k}}_i$ *is an isomorphism for all* $i\leqslant j$, *iv)* $\mathcal{W}^{\mathfrak{k}}_j = \oplus_{i=1}^j M^{\mathfrak{k}}_i$.

Proof It can be proved that, for Levi $\mathfrak{k}$, both $M^{\mathfrak{k}}_j$ and $\mathrm{gr}\,\mathcal{V}^{\mathfrak{k}}_j$ are parabolically induced from the same $U_q(\mathfrak{k})$-module. Hence the map $M^{\mathfrak{k}}_j \to \mathrm{gr}\,\mathcal{V}^{\mathfrak{k}}_j$ is epimorphism and isomorphism simultaneously unless it is zero.

The implication ii) $\Rightarrow$ i) is trivial. With $\mathcal{W}^{\mathfrak{k}}_1 = \mathcal{V}^{\mathfrak{k}}_1$, suppose that ii) is violated and let $k>1$ be the smallest such that $\mathcal{W}^{\mathfrak{k}}_k \neq \mathcal{V}^{\mathfrak{k}}_k$. Comparison of weight subspaces gives $\dim\mathcal{V}^{\mathfrak{k}}_j[\lambda+\nu_j] = \dim\mathcal{W}^{\mathfrak{k}}_j[\lambda+\nu_j]+1$ for all $j\geqslant k$, thus i) $\Rightarrow$ ii). Assuming ii) we find that all maps $M^{\mathfrak{k}}_i \to \mathrm{gr}\,\mathcal{V}^{\mathfrak{k}}_i$ are surjective and therefore injective; hence iv). Conversely, iv) implies that all maps $\mathcal{W}^{\mathfrak{k}}_i \to \mathrm{gr}\,\mathcal{V}^{\mathfrak{k}}_i$ are surjective. Since, $\mathcal{W}^{\mathfrak{k}}_1 = \mathcal{V}^{\mathfrak{k}}_1$, induction on i then proves ii). Furthermore, iv) implies that $M^{\mathfrak{k}}_i \to \mathrm{gr}\,\mathcal{V}^{\mathfrak{k}}_i$ are isomorphisms, and then $M^{\mathfrak{k}}_i \cap \mathcal{W}^{\mathfrak{k}}_{i-1} \subset M_i\cap\mathcal{V}^{\mathfrak{k}}_{i-1} = \{0\}$, which proves iii). Finally, induction on i yields iii) $\Rightarrow$ ii). □

For any $\mathfrak{k}$, a direct sum decomposition $V\otimes M^{\mathfrak{k}}_\lambda = \oplus_j M^{\mathfrak{k}}_j$ implies that the operator $\mathcal{Q}$ is semisimple on $V\otimes M^{\mathfrak{k}}_\lambda$. More generally, $\mathcal{Q}$ is semisimple if $V\otimes M^{\mathfrak{k}}_\lambda = \mathcal{W}^{\mathfrak{k}}_7$. That is the case if all maps $\wp^{\mathfrak{k}}_j\colon M^{\mathfrak{k}}_j \to \mathrm{gr}\,\mathcal{V}^{\mathfrak{k}}_j$ are onto, i.e. the generators of $M^{\mathfrak{k}}_j$ are not killed by $\wp^{\mathfrak{k}}_j$.

6 Module Structure of $V \otimes M_\lambda$

In this section, M_λ is the Verma module of highest weight λ. We work out exact criteria for decomposition of $V \otimes M_\lambda$ into a direct sum of submodules of highest weight. To that end, we undertake a detailed study of singular vectors $\hat{u}_j = \hat{F}(w_j \otimes v_\lambda) \in V \otimes M_\lambda$ as rational trigonometric functions $\mathfrak{h}^* \to V \otimes U_q(\mathfrak{g}_-)$, upon the natural identification of M_λ with $U_q(\mathfrak{g}_-)$ as vector spaces. Although they cannot be evaluated at poles, singular vectors are defined up to a scalar multiplier and can be regularized. We end up with rescaled singular vectors u_j, $j \in I$, that are regular on $\mathfrak{h}^*$ and never turn zero.

6.1 Singular Vectors in $V \otimes M_\lambda$

The vectors $\hat{u}_j = \hat{F}(w_j \otimes v_\lambda)$, $j \in I$, are expanded as

$$\hat{u}_j = \sum_{i=1}^{j} w_i \otimes \hat{f}_{ij} v_\lambda \in V \otimes M_\lambda. \tag{6.14}$$

They are singular for all λ where defined and generate submodules $M_j \subset V \otimes M_\lambda$ of highest weight $\lambda + \nu_j$. They have rational trigonometric dependence on λ and may have zeros and poles. As singular vectors matter up to scalar factors, it is convenient to pass from $\hat{u}_j$ to $\check{u}_j = \bar{A}^j_{1,\ldots,j-1}(\lambda)\hat{u}_j$, which are regular in λ. Then

$$\check{u}_j = \sum_{i=1}^{j} w_i \otimes \check{u}_{ij}, \quad \text{where} \quad \check{u}_{ij} = \check{f}_{ij} v_\lambda, \quad \check{f}_{ij} = \hat{f}_{ij} \bar{A}^j_{i,\ldots,j-1} \in U_q(\mathfrak{b}_-). \tag{6.15}$$

Each $\check{u}_j$ generates a submodule $M_j \subset V \otimes M_\lambda$ if does not turn zero, otherwise $\check{u}_j$ needs rescaling. That is the subject of our further study.

Remark that, for any $U_q(\mathfrak{g})$-module Z, a singular vector $u = \sum_{i\in I} w_i \otimes z_i \in V \otimes Z$, defines a $U_q(\mathfrak{g}_+)$-equivariant map $V^* \to Z$. Since the $U_q(\mathfrak{g}_+)$-module V^* is cyclicly generated by z_1, we call it generating coefficient of u.

Lemma 6.1 *Suppose that Z is generated by the highest weight vector v_λ. Suppose that $\check{f}_{mj} v_\lambda = 0$ for some $m < j$. Then $\check{f}_{ij} v_\lambda = 0$ for all $i \leqslant m$.*

Proof It follows now from Eq. 5.12 that

$$\begin{aligned} \check{f}_{m-1,j} v_\lambda &= \bar{A}^j_{m-1,\ldots,j-1}(\lambda) f_{m-1,j} v_\lambda + a_m(\lambda) f_{m-1,m} \check{f}_{mj} v_\lambda \\ &\quad + \sum_{k=m+1}^{j} a_k(\lambda) f_{m-1,k} \left(\bar{A}^j_{m,\ldots k-1}(\lambda) \check{f}_{kj}\right) v_\lambda, \end{aligned}$$

where $a_i(\lambda)$ are numerical factors. The second term disappears by the hypothesis. Let us show that the other terms are zero too. Recall that for all $k \geqslant m$ and $\mu = \nu_k - \nu_{k+1}$, $e_\mu \check{f}_{kj} = \check{f}_{k+1,j} \bar{A}^j_k \mod U_q(\mathfrak{g})\mathfrak{g}_+$, cf. [18]. Then $\check{f}_{mj} v_\lambda = 0$ implies $\bar{A}^j_{m,\ldots,k-1}(\lambda)\check{f}_{kj} v_\lambda = 0$ for all $k \geqslant m+1$. Firstly this kills the sum. Furthermore, setting $k = j$ we derive $\bar{A}^j_{m,\ldots,j-1}(\lambda) = 0$. Since $\bar{A}^j_{m,\ldots,j-1} \sqsubset \bar{A}^j_{m-1,\ldots,j-1}$, the first term is zero, which proves $\check{f}_{m-1,j} v_\lambda = 0$. Descending induction on i completes the proof for all $i \leqslant m$. ☐

Corollary 6.2 *The vector $\check{u}_j \in V \otimes Z$ turns zero i) only if $\bar{A}^j_m(\lambda) = 0$, ii) if only if $\check{f}_{mj} v_\lambda = 0$ for some $m < j$.*

Proof "Only if" in i) and ii) follow from the equalities $\check{u}_{jj} = \bar{A}^j_{1,\ldots,j-1} v_\lambda$ and, respectively, $\check{u}_{1j} = \check{f}_{1j} v_\lambda$. "If" in ii) is due to Lemma 6.1 because then the generating coefficient $\check{u}_{1j} = \check{f}_{1j} v_\lambda$ vanishes. □

Remark 6.3 Suppose that Z is a family of $U_q(\mathfrak{g})$-modules of highest weight λ ranging in an algebraic set $\Theta \subset \mathfrak{h}^*$. Then Lemma 6.1 admits an obvious modification if one replaces equality to zero with divisibility by some $\phi \in \mathbb{C}_q[\Theta]$. Assuming it indecomposable, Corollary 6.2 can be appropriately reworded if $\mathbb{C}_q[\Theta]$ is a unique factorization domain and Z has no zero divisors. In what follows, we apply this modification to $\Theta = \Theta^{\mathfrak{k}}$ and $Z = M^{\mathfrak{k}}_\lambda$.

In the next statement we essentially assume that $\mathfrak{k} \neq \mathfrak{h}$.

Proposition 6.4 *The submodule $M^{\mathfrak{k}}_j \subset V \otimes M^{\mathfrak{k}}_\lambda$ vanishes for all $j \in \bar{I}^{\mathfrak{k}}$.*

Proof It is sufficient to consider the case $\Pi_{\mathfrak{k}} = \{\mu\}$. Choose $\lambda \in \Theta^{\mathfrak{k}}$ so that $\check{u}_j \neq 0$. By Proposition 6.8 below, such weights are dense in $\Theta^{\mathfrak{k}}$. Let $i \in I$ be such that $\nu_i - \nu_j = \mu$. As $\bar{A}^j_i(\lambda) = 0$, for all $k \leqslant i$ one has

$$\check{f}_{kj} v_\lambda \simeq \sum_{k<\vec{m}<i} f_{k,\vec{m},i} \left(A^j_{k,\vec{m}}(\lambda) \bar{A}^j_{k,\ldots,i-1}(\lambda) \right) (\check{f}_{ij} v_\lambda) \in M_{\lambda-\mu} \subset M_\lambda.$$

On the other hand, $\check{u}_{kj} = \bar{A}^j_{1,\ldots,k-1}(\lambda) \check{f}_{kj} v_\lambda = 0$ for all $i < k$ because the numerical factor contains vanishing $\bar{A}^j_i(\lambda)$. Therefore $\check{u}_j \in V \otimes M_{\lambda-\mu}$ for all $\lambda \in \Theta^{\mathfrak{k}}$ and vanishes in $V \otimes M^{\mathfrak{k}}_\lambda$, along with $M^{\mathfrak{k}}_j$. □

6.2 Projection to gr $\mathcal{V}_\bullet$

It follows from Proposition 5.3 that the image of $\hat{u}_j$ in gr $\mathcal{V}_\bullet$ lies in gr $\mathcal{V}_j$, and thus $\hat{u}_j = \hat{D}_j w^j_\lambda \mod \mathcal{V}_{j-1}$ with some $\hat{D}_j \in \mathbb{C}$. Up to a scalar factor, $\prod_{j=1}^7 \hat{D}_j$ coincides with the determinant of the extremal twist operator $\theta_{V,\hat{M}^{\mathfrak{h}}_\lambda}$, [19]. Its inverse was calculated in [9] in connection with dynamical Weyl group. Obviously $\hat{D}_1 = 1$. For higher j, applying the results of [9], one has

$$\hat{D}_2 \simeq \frac{[\xi_{12}]_q}{[\eta_{12}]_q}, \quad \hat{D}_3 \simeq \frac{[\xi_{13}]_q}{[\eta_{13}]_q}\frac{[\xi_{23}]_q}{[\eta_{23}]_q}, \quad \hat{D}_4 \simeq \frac{[\xi_{14}]_q}{[\eta_{14}]_q}\frac{[\xi_{24}]_q}{[\eta_{24}]_q}\frac{[\xi_{34}]_q}{[\eta_{34}]_q}, \quad \hat{D}_5 \simeq \frac{[\xi_{15}]_q}{[\eta_{15}]_q}\frac{[\xi_{25}]_q}{[\eta_{25}]_q}\frac{[\frac{\xi_{35}}{2}]_q}{[\frac{\eta_{35}}{2}]_q},$$

$$\hat{D}_6 \simeq \frac{[\xi_{16}]_q}{[\eta_{16}]_q}\frac{[\xi_{36}]_q}{[\eta_{36}]_q}\frac{[\frac{\xi_{26}}{2}]_q}{[\frac{\eta_{26}}{2}]_q}\frac{[\xi_{56}]_q}{[\eta_{56}]_q}, \quad \hat{D}_7 \simeq \frac{[\xi_{27}]_q}{[\eta_{27}]_q}\frac{[\xi_{37}]_q}{[\eta_{37}]_q}\frac{[\frac{\xi_{17}}{2}]_q}{[\frac{\eta_{17}}{2}]_q}\frac{[\xi_{57}]_q}{[\eta_{57}]_q}\frac{[\xi_{67}]_q}{[\eta_{67}]_q},$$

where the quantities

$$\xi_{ij} = h_i - h_j + \rho_i - \rho_j + \frac{1}{2}(||\nu_i||^2 - ||\nu_j||^2) \in \mathfrak{h} + \mathbb{C}$$

are related to eigenvalues of the operator $\mathcal{Q}$ by $q^{2\xi_{ij}} = x_i \bar{x}_j$. For reader's convenience, ξ_{ij} are listed in Appendix A.1.

Let $\Theta_j \subset \mathfrak{h}^*$, $j > 4$, denote the set of weights such that $q^{2\eta_{4j}(\lambda)} = -q^2$.

Lemma 6.5 *The module M_j is not contained in $\mathcal{V}_{j-1}$ for a dense open subset $\Theta_j^\circ \subset \Theta_j$.*

Proof Observe that $q^{2\xi_{4j}} = -q^2 \neq 1$ and $q^{2\xi_{j'j}} = q^8 \neq 1$. For all other $i < j$, the functions $q^{2\xi_{ij}}$ are not constant on Θ_j. Therefore all $q^{2\xi_{ij}}$ with $i < j$ are distinct from 1, off an algebraic subset in Θ_j. For such weights, M_j cannot be in $\mathcal{V}_{j-1}$, since the $\mathcal{Q}$-eigenvalue x_j is distinct from the $\mathcal{Q}$-eigenvalues $\{x_i\}_{i=1}^{j-1}$ on $\mathcal{V}_{j-1}$. □

Corollary 6.6 *For all $j > 4$, $\check{f}_{1j}$ identically vanishes on Θ_j.*

Proof The product $\check{D}_j = \prod_{k=1}^{j-1} \bar{A}_k^j \hat{D}_j$ is polynomial in $q^{\pm h_\mu}$ and turns zero on Θ_j for $j > 4$ thanks to the factor $\bar{A}_{j'}^j$ and the equality $\frac{\eta_{j'j}}{2} = \eta_{4j} - 1$. The tensor $\check{u}_j \in V \otimes M_\lambda$ is projected to $\check{D}(\lambda) w_\lambda^j = 0 \mod \mathcal{V}_{j-1}$. That is possible only in the following two cases: either $\check{u}_j$ turns zero or $M_j \subset \mathcal{V}_{j-1}$. By Lemma 6.5, $\check{u}_j(\lambda) = 0$ on Θ_j° and hence on Θ_j. This yields $\check{u}_{1j} = \check{f}_{1j} v_\lambda = 0$ on Θ_j for the generating coefficient. □

6.3 Regularization of Singular Vectors in $V \otimes M_\lambda$

In this section, we evaluate a scalar function $\delta_j \sqsubset \check{u}_j$ and show that renormalized singular vectors $\delta_j^{-1} \check{u}_j$ do not turn zero at all λ.

Denote by J the two-sided ideal in $U_q(\mathfrak{g}_-)$ generated by the relation $f_\alpha f_\beta = \bar{q}^3 f_\beta f_\alpha$. The non-zero elements f_{ij} modulo J read

$$f_{12} = f_\alpha, \quad f_{23} = [3]_q f_\beta, \quad f_{34} = f_\alpha, \quad f_{45} = f_\alpha, \quad f_{56} = [3]_q f_\beta, \quad f_{67} = f_\alpha,$$

$$f_{24} = (\bar{q}^3 - q^3) f_\beta f_\alpha, \quad f_{35} = \frac{\bar{q}^2 - 1}{[2]_q} f_\alpha^2, \quad f_{57} = (\bar{q}^3 - q^3) f_\beta f_\alpha,$$

$$f_{25} = \frac{(\bar{q}^3 - q)(\bar{q}^3 - q^3)}{[2]_q^2} f_\beta f_\alpha^2.$$

Introduce $\check{g}_{ij}$ as polynomials in $y_1^{\pm 1}, \ldots, y_7^{\pm 1}$ with coefficients in $U_q(\mathfrak{g}_-)$ by similar formulas as $\check{f}_{ij}$ with all $\bar{A}_k^j$ in Eqs. 5.12 and 6.15 replaced by $\bar{A}_k = \frac{1 - y_k}{q - \bar{q}}$.

Lemma 6.7 *One has $\check{g}_{35} \simeq f_\alpha^2 \frac{\bar{q} y_4 + q}{q + \bar{q}}$. Furthermore, $\check{g}_{25} \simeq [3]_q f_\beta f_\alpha^2 \frac{\bar{q} y_4 + q}{q + \bar{q}} \mod J$.*

Proof All calculations will be done modulo J. First of all,

$$\check{g}_{35} \simeq f_{35} \bar{A}_4 + f_{34} f_{45} = f_\alpha^2 \left(\frac{\bar{q}^2 - 1}{[2]_q} \frac{y_4 - 1}{\bar{q} - q} + 1 \right) = f_\alpha^2 \frac{\bar{q} y_4 + q}{q + \bar{q}}.$$

Substitute $f_{ij} \mod J$ into $\check{g}_{25} \simeq f_{25}\bar{A}_3\bar{A}_4 + f_{23}f_{35}\bar{A}_4 + f_{24}f_{45}\bar{A}_3 + f_{23}f_{34}f_{45}$ and get

$$\check{g}_{25} \simeq f_\beta f_\alpha^2 \left(\frac{(\bar{q}^3 - q)(\bar{q}^3 - q^3)}{[2]_q^2} \bar{A}_3\bar{A}_4 + [3]_q \frac{\bar{q}^2 - 1}{[2]_q} \bar{A}_4 + (\bar{q}^3 - q^3)\bar{A}_3 + [3]_q \right).$$

Computation of the coefficient in the brackets completes the proof. □

Set $\delta_j = \frac{q^{\eta_{4j}-1} + q^{-\eta_{4j}+1}}{q+\bar{q}} = \frac{q^{\frac{\eta_{j'j}}{2}} + q^{-\frac{\eta_{j'j}}{2}}}{q+\bar{q}}$ for $5 \leqslant j$ and $\delta_j = 1$ for $j = 1, 2, 3, 4$. Corollary 6.6 assures that $\check{f}_{1j}/\delta_j$ is a polynomial in $q^{\pm h_\mu}$ via identification $y_i = q^{2\eta_{ij}}$, $i < j$.

Proposition 6.8 *For all $j > 1$, the vectors $\check{f}_{1j}/\delta_j$ do not turn zero on $\mathfrak{h}^*$.*

Proof Again, all calculations are done modulo J. The statement is trivial for $j = 2$. For $j = 3$, it follows from the factorization $\check{f}_{13} = f_{12}f_{23}$. We also have

$$\check{f}_{14} \simeq f_{12}\check{f}_{24} \simeq f_\alpha([3]_q f_\beta f_\alpha + (\bar{q}^3 - q^3) f_\beta f_\alpha \bar{A}_3^4) = [3]_q f_\alpha f_\beta f_\alpha q^{\eta_{35}} \neq 0,$$

which proves the case $j = 4$.

For each $j \geqslant 5$ and all $i < j$ we assign $y_i = q^{2\eta_{ij}}$ and use Lemma 6.7: the key point is that δ_j cancels the factor $\bar{q}y_4 + q$ in all cases. Modulo J, we have $\check{f}_{15}(\lambda) \simeq f_{12}\check{g}_{25}$ and $\check{f}_{16}(\lambda) \simeq f_{12}\check{g}_{25}f_{56}$. This implies the statement for $j = 5, 6$. Finally,

$$\check{f}_{17}(\lambda) \simeq f_{12}\check{g}_{27} \simeq f_{12}(\check{g}_{26}f_{67} + \check{g}_{25}f_{57}\bar{A}_6) \simeq f_{12}(\check{g}_{25}f_{56}f_{67} + \check{g}_{25}f_{57}\bar{A}_6) = f_\alpha \check{g}_{25} f_\beta f_\alpha [3]_q y_6,$$

which proves it for $j = 7$. □

6.4 Decomposition of $V \otimes M_\lambda$

Denote by u_j the singular vectors $\check{u}_j/\delta_j(\lambda)$ for all $j \in I$. Then $u_j = D_j w_j^\lambda \mod \mathcal{V}_{j-1}$ with $D_j \simeq \prod_{i=1}^{j-1} \phi_{ij}$, where $\phi_{ij} = [\xi_{ij}]_q$ if $i \neq j'$ and $\phi_{j'j} = [\frac{\xi_{j'j}}{2}]_q$.

Lemma 6.9 *The submodule M_j is contained in M_i with $i < j$ if and only if $\phi_{ij}(\lambda) = 0$.*

Proof The eigenvalues $q^{2(\rho,\nu_1)}x_j$ of the operator $q^{2(\rho,\nu_1)}\mathcal{Q}$ on the submodules M_j are

$$q^{2(\lambda,2\alpha+\beta)+10},\ q^{2(\lambda,\alpha+\beta)+8},\ q^{2(\lambda,\beta)+2},\ q^{-2},\ q^{-2(\lambda,\beta)-2},\ q^{-2(\lambda,\alpha+\beta)-8},\ q^{-2(\lambda,2\alpha+\beta)-10}, \tag{6.16}$$

counting from the left. If $\phi_{ij}(\lambda)$ and hence $D_j(\lambda)$ turns zero, then $x_j = x_i$ and $M_j \subset \mathcal{V}_{j-1}$, by Proposition 5.5. Suppose that x_j is distinct from x_k if $k < j$ and $k \neq i$. As follows from Eq. 6.16, such weights are dense in the set of solutions to $\phi_{ij}(\lambda) = 0$. Then $\mathcal{W}_{j-1} = \mathcal{V}_{j-1}$, and M_j can lie only in M_i and hence it does for all such weights.

Conversely, let λ be such that $M_j \subset M_i$. Then $[\xi_{ij}]_q = 0$ and hence $\phi_{ij}(\lambda) = 0$ if $i \neq j'$. If $i = j'$, we can assume that $M_j \not\subset M_k$ for $k \neq j'$ since λ is in the closure of such weights, as follows from Eq. 6.16. Proposition 5.5 then suggests that $D_j(\lambda) = 0$, and the only vanishing factor can be $\phi_{j'j}$. Then it is true for all λ. □

As an application of Lemma 6.9, we describe direct sum decomposition of the module $V\otimes M_\lambda$. This corresponds to maximal conjugacy classes, with the stabilizer subalgebra $\mathfrak{k}=\mathfrak{h}$. Fix $t=\mathrm{diag}(t_i)\in T$ and choose λ to fulfill the condition $tq^{-2h_\rho}=q^{2h_\lambda}$. This fixes the relation between the entries of t and the $\mathcal{Q}$-eigenvalues as $t_i=q^{2(\lambda+\rho,\nu_i)}=x_iq^{2\delta_{i4}+2(\rho,\nu_1)}$.

Proposition 6.10 *Let a point $t\in T^{\mathfrak{h}}$ be parameterized as in* Eqs. 2.2 *and* 2.3 *and the weight $\lambda=\lambda(t)$ chosen as prescribed above. Then $V\otimes M_\lambda=\oplus_{i=1}^7 M_i$ if and only if $q^{-2}\neq x,y,xy$ for regular t and $q^{-4}\neq x^2$ for borderline t.*

Proof The sum $\sum_{i=1}^7 M_i$ exhausts all of $V\otimes M_\lambda$ if and only if it is direct, by Proposition 5.5 or, equivalently, if $D^{\mathfrak{h}}=\prod_{i=1}^7 D_i$ is not zero. Explicitly,

$$D^{\mathfrak{h}}\simeq\prod_{\substack{i<j\\ i\neq j',4}}(x_i-x_j)\prod_{i=1}^{3}(x_i-q^{2\rho_1})\simeq\prod_{\substack{i<j\\ i\neq j'\\ i,j\neq 4}}(t_i-t_j)\prod_{i=1}^{3}(t_i-q^{-2})\prod_{i=1}^{3}(t_i-1)\simeq\prod_{i=1}^{3}(t_i-q^{-2})$$

for $t\in T^{\mathfrak{h}}$. This implies the stated conditions on q guaranteeing $D^{\mathfrak{h}}\neq 0$ (mind that q is not a root of unity). $\square$

Next we essentially assume that $\mathfrak{k}\neq\mathfrak{h}$ and define $\phi_j^{\mathfrak{k}}=\prod_{i\in\bar{I}_j^{\mathfrak{k}}}\phi_{ij}$, $j\in I^{\mathfrak{k}}$.

Lemma 6.11 *For every $j\in I^{\mathfrak{k}}$, the vector $\pi_\lambda^{\mathfrak{k}}(u_j)\in V\otimes M_\lambda^{\mathfrak{k}}$ vanishes once $\phi_j^{\mathfrak{k}}(\lambda)=0$. If $\mathfrak{k}$ is of type $\mathfrak{k}_s$ or $\mathfrak{k}_l$, then $\pi_\lambda^{\mathfrak{k}}(u_j)$ is divisible by $\phi_j^{\mathfrak{k}}$.*

Proof By Lemma 6.9, $u_j\in M_i$ for $i\in\bar{I}_j^{\mathfrak{k}}$ once $\phi_j^{\mathfrak{k}}(\lambda)=0$. On the other hand, $\pi_\lambda^{\mathfrak{k}}(u_i)=0$ if $\lambda\in\Theta^{\mathfrak{k}}$, by Proposition 6.4. Therefore $\pi_\lambda^{\mathfrak{k}}(u_j)$ vanishes in $V\otimes M_\lambda^{\mathfrak{k}}$. If the semisimple part of $\mathfrak{k}$ has rank 1, trigonometric polynomials on $\Theta^{\mathfrak{k}}$ form a principal ideal domain as $\dim\Theta^{\mathfrak{k}}=1$. Therefore $\pi_\lambda^{\mathfrak{k}}(u_j)$ is divisible by $\phi_j^{\mathfrak{k}}(\lambda)$. $\square$

7 Module Structure of $V\otimes M_\lambda^{\mathfrak{k}_s}$

Throughout this section $\nu\in\mathrm{R}^+$ is a short root and $\mathfrak{k}=\mathfrak{k}_s$ is the reductive subalgebra of maximal rank with the root system $\{\pm\nu\}$. We aim to prove that the vectors $u_j^{\mathfrak{k}}=\frac{1}{\phi_j^{\mathfrak{k}}(\lambda)}\pi_\lambda^{\mathfrak{k}}(u_j)\in V\otimes M_\lambda^{\mathfrak{k}}$ are regular functions of λ and do not vanish at all weights. Their projections to $\mathrm{gr}\,\mathcal{V}_\bullet^{\mathfrak{k}}$ are equal to $D_j^{\mathfrak{k}}w_j^\lambda$ with $D_j^{\mathfrak{k}}\simeq\prod_{i\in I_j^{\mathfrak{k}}}\phi_{ij}$.

7.1 Regularization of Singular Vectors in $V\otimes M_\lambda^{\mathfrak{k}}$

Assuming $j\in I^{\mathfrak{k}}$, denote by $c_j^{\mathfrak{k}}$ the coefficient in the expansion $u_j^{\mathfrak{k}}\simeq w_j\otimes c_j^{\mathfrak{k}}v_\lambda+\ldots$, where the suppressed terms belong to $\sum_{i<j}w_i\otimes M_\lambda^{\mathfrak{k}}$. Up to a non-vanishing factor, $c_j^{\mathfrak{k}}=\frac{\prod_{i<j}[\eta_{ij}]}{d_j\phi_j^{\mathfrak{k}}}$.

Table 1 Type $\mathfrak{k} \simeq \mathfrak{k}_s$

$j \in I_{\mathfrak{k}}$	$\phi_j^{\mathfrak{k}}$	$D_j^{\mathfrak{k}}$	$c_j^{\mathfrak{k}}$
$\nu = \alpha$, $(\lambda, \alpha) = 0$, $\theta = (\lambda, \beta)$			
1	1	1	1
3	$[\theta + 3]_q$	$[\theta + 3]_q$	$[\theta]_q$
6	$[\theta + 4]_q[\theta + 3]_q[\theta + 3]_q$	$[2\theta + 9]_q[\theta + 5]_q$	$[\theta + 2]_q[2\theta + 6]_q$
$\nu = \alpha + \beta$, $q^{2(\lambda,\nu)+6} = 1$, $\theta = (\lambda, \alpha)$			
1	1	1	1
2	1	$[\theta + 1]_q$	$[\theta]_q$
5	$[\theta + 1]_q[\theta]_q$	$[2\theta + 3]_q[\theta + 2]_q$	$[\theta - 1]_q[2\theta]_q$
$\nu = 2\alpha + \beta$, $q^{2(\lambda,\nu)+8} = 1$, $\theta = (\lambda, \alpha)$			
1	1	1	1
2	1	$[\theta + 1]_q$	$[\theta]_q$
3	1	$[\theta]_q[2\theta + 1]_q$	$[\theta + 1]_q[2\theta + 4]_q$

Proposition 7.1 *The vectors $u_j^{\mathfrak{k}}$ do not vanish at all weights.*

Proof We parameterize $\Theta^{\mathfrak{k}} \subset \mathfrak{h}^*$ with the complex variable θ as in Table 1.

The case $\nu = \alpha$ follows from the fact that the nil-sets of $D_j^{\mathfrak{k}}$ and $c_j^{\mathfrak{k}}$ do not intersect, cf. Table 1.

In the case of $\nu = \alpha + \beta$, the statement is trivial for $j = 1$ and immediate for $u_2^{\mathfrak{k}} \simeq -q^{-1} w_1 \otimes f_\alpha v_\lambda + w_2 \otimes [\theta]_q v_\lambda$. Let us prove it for $j = 5$. Choose a basis $f_\beta f_\alpha^3 v_\lambda$, $f_\alpha f_\beta f_\alpha^2 v_\lambda$, $f_\alpha^2 f_\beta f_\alpha v_\lambda$ in the weight subspace $M_\lambda^{\mathfrak{k}}[\lambda - \beta - 3\alpha]$. The coefficient in $\check{f}_{15}/(\delta_5 \phi_5^{\mathfrak{k}})$ corresponding to $f_\beta f_\alpha^3$ is equal to $[\eta_{25}]_q = [\theta + 1]_q$, up to a non-vanishing factor. Hence $u_5^{\mathfrak{k}}$ does not turn zero unless $[\theta + 1]_q = 0$. However, $D_5^{\mathfrak{k}} \neq 0$ at such λ. Therefore $u_5^{\mathfrak{k}} \neq 0$ at all weights from $\Theta^{\mathfrak{k}}$.

Finally, consider the case $\nu = 2\alpha + \beta$. The projection $M_\lambda \to M_\lambda^{\mathfrak{k}}$ is an isomorphism on subspaces of weights $\lambda - \mu$ with $\mu < 2\alpha + \beta$. Such are the weights of the generating coefficients in $u_j^{\mathfrak{k}} = \pi_\lambda^{\mathfrak{k}}(u_j)$. They do not turn zero as $u_j \neq 0$ at all weights. This completes the proof. □

7.2 Decomposition of $V \otimes M_\lambda^{\mathfrak{k}_s}$

Fix $t \in T^\nu$ and let $\mathfrak{k} \simeq \mathfrak{k}_s$ denote the stabilizer of t. Choose $\lambda \in \Theta^{\mathfrak{k}}$ from the equality $q^{2\lambda+2\rho} = tq^\nu$. We use $x \in \mathbb{C}^\star$, $x^2 \neq 1$, to parameterise the spectrum $\{x^{\pm 1}, 1\}$ of t as in Eq. 2.4.

Proposition 7.2 *The module $V \otimes M_\lambda^{\mathfrak{k}}$ splits into the direct sum $\oplus_{j \in I^{\mathfrak{k}}} M_j^{\mathfrak{k}}$ if and only if $q \neq x^{\pm 1}$.*

Proof Table 1 gives $D^{\mathfrak{k}} = \prod_{j \in I^{\mathfrak{k}}} D_j^{\mathfrak{k}} \simeq (q - x)(q - x^{-1})$. The sum $\sum_{j \in I^{\mathfrak{k}}} M_j^{\mathfrak{k}}$ exhausts all of $V \otimes M_\lambda^{\mathfrak{k}}$ if and only if $D^{\mathfrak{k}}$ does not vanish. It follows that the eigenvalues $qx^{\pm 1}$, q^2 of the operator $q^{10}\mathcal{Q}$ on $V \otimes M_\lambda^{\mathfrak{k}}$ are pairwise distinct, hence the sum is direct. □

8 Module Structure of $V \otimes M_\lambda^{\mathfrak{k}_l}$

In this section, $\nu \in \mathrm{R}^+$ is one of the three long roots, and $\mathfrak{k} \simeq \mathfrak{k}_l \subset \mathfrak{g}$ is the reductive subalgebra of maximal rank with the root system $\{\pm\nu\}$. Now $\Theta^{\mathfrak{k}}$ is the set of weights λ that satisfy the condition $q^{2(\lambda+\rho,\nu)-6} = 1$. We parameterize it with the complex variable $\theta = (\lambda, \alpha)$. There are two pairs $(l, k) \in P(\nu)$, so $\#\bar{I}_{\mathfrak{k}} = 2$ and $\#I_{\mathfrak{k}} = 5$.

8.1 Regularization of Singular Vectors for Quasi-Levi $\mathfrak{k}$

The case of quasi-Levi $\mathfrak{k}_l$ turns out to be simpler, so we consider it first. For $\nu = 3\alpha+2\beta$, we have $I^{\mathfrak{k}} = \{1, 2, 3, 4, 5\}$, $q^{2(\lambda,3\alpha+2\beta)+12} = 1$, and $\phi_j^{\mathfrak{k}} = 1$ for all $j \in I^{\mathfrak{k}}$. For $\nu = 3\alpha+\beta$, we have $I^{\mathfrak{k}} = \{1, 2, 3, 4, 6\}$, $q^{2(\lambda,3\alpha+\beta)+6} = 1$, and $\phi_j^{\mathfrak{k}} = 1$ for $j \in I^{\mathfrak{k}}$ apart from $\phi_6^{\mathfrak{k}} = [3\theta]_q$, where $\theta = (\lambda, \alpha)$.

For each $\mu < \nu$, the projection $M_\lambda[\lambda - \mu] \to M_\lambda^{\mathfrak{k}}[\lambda - \mu]$ is an isomorphism. Therefore $\pi_\lambda^{\mathfrak{k}}(u_j) = u_j^{\mathfrak{k}}$ does not turn zero unless, maybe, for $\nu = 3\alpha + \beta$, $j = 6$. In the latter case $D_6^{\mathfrak{k}} = [3\theta-3]_q[2\theta-1]_q[\theta-2]_q[2\theta]_q$ and $c_6^{\mathfrak{k}} = [2\theta+1]_q[\theta-1]_q[2\theta]_q[3\theta+3]_q$, so $u_6^{\mathfrak{k}}$ may vanish only at $q^{2\theta} = \pm 1, q^2$. However, $u_{36}^{\mathfrak{k}} \simeq [2\theta+1]_q \check{f}_{36} v_\lambda$, and one can easily check that

$$\check{f}_{36} = \frac{[[f_\beta, f_\alpha]_q, f_\alpha]_{q^3}}{[2]_q^2} \bar{A}_4\bar{A}_5 + f_\alpha[f_\alpha, f_\beta]_{q^3}\bar{A}_5 + f_\alpha^2 f_\beta \left(\frac{q+q^{-4\theta-1}}{[2]_q}\right)[3]_q \neq 0$$

at all weights. Hence $u_6^{\mathfrak{k}} \neq 0$ at all weights.

8.2 Regularization of Singular Vectors for Levi $\mathfrak{k}$

From Table 2 we conclude that $u_j/\phi_j^{\mathfrak{k}}$ may be divisible by the following factors:

1. $[\theta + 3]_q \sqsubset \bar{A}_2^4$, for $j = 4$,
2. $[\theta + 2]_q \sqsubset \bar{A}_2^5$, $[\theta]_q \sqsubset \bar{A}_4^5$, for $j = 5$,
3. $[2\theta + 4]_q \sqsubset \bar{A}_4^7$, $[\theta + 3]_q \sqsubset \bar{A}_5^7$, for $j = 7$.

Introduce $\psi_j^{\mathfrak{k}}$ for $j \in I^{\mathfrak{k}}$ as

$$\psi_1^{\mathfrak{k}} = 1, \quad \psi_2^{\mathfrak{k}} = 1, \quad \psi_4^{\mathfrak{k}} = [\theta+3]_q, \quad \psi_5^{\mathfrak{k}} = [\theta+2]_q, \quad \psi_7^{\mathfrak{k}} = [\theta+2]_q[\theta+3]_q.$$

Proposition 8.1 *For all $j \in I^{\mathfrak{k}}$, the singular vector $\pi_\lambda^{\mathfrak{k}}(u_j)/\phi_j^{\mathfrak{k}}$ is divisible by $\psi_j^{\mathfrak{k}}$, and $u_j^{\mathfrak{k}} = \pi_\lambda^{\mathfrak{k}}(u_4)/(\phi_j^{\mathfrak{k}}\psi_j^{\mathfrak{k}}) \neq 0$ at all λ.*

Table 2 $\nu = \beta$, $(\lambda, \beta) = 0$, $(\lambda, \alpha) = \theta$

$j \in I_{\mathfrak{k}}$	$\phi_j^{\mathfrak{k}}$	$\check{D}_j/(\phi_j^{\mathfrak{k}}\delta_j)$	$\check{c}_j/(\phi_j^{\mathfrak{k}}\delta_j)$	$D_j^{\mathfrak{k}}$
1	1	1	1	1
2	1	$[\theta+1]_q$	$[\theta]_q$	$[\theta+1]_q$
4	$[\theta+2]_q$	$[2\theta+6]_q[\theta+5]_q$	$\frac{[2\theta+4]_q}{[\theta+2]_q}[\theta+3]_q[\theta]_q$	$\frac{[2\theta+6]_q}{[\theta+3]_q}[\theta+5]_q$
5	$[\theta+1]_q$	$[3\theta+6]_q[2\theta+5]_q[\theta]_q$	$\frac{[3\theta+3]_q}{[\theta+1]_q}[\theta-1]_q[2\theta+4]_q[\theta]_q$	$\frac{[3\theta+6]_q}{[\theta+2]_q}[2\theta+5]_q[\theta]_q$
7	$[3\theta+6]_q[\theta+1]_q$	$[2\theta+5]_q[3\theta+9]_q$ $\times[2\theta+4]_q[\theta+4]_q$	$[2\theta+3]_q\frac{[3\theta+3]_q}{[\theta+1]_q}[2\theta+4]_q$ $\times[\theta+3]_q[\theta]_q$	$[2\theta+5]_q\frac{[3\theta+9]_q}{[\theta+3]_q}$ $\times\frac{[2\theta+4]_q}{[\theta+2]_q}[\theta+4]_q$

Proof There is nothing to prove for $j = 1, 2$, so we assume $j = 4, 5, 7$. We should only check divisibility, because "$u_j^{\mathfrak{k}} \neq 0$" then follows from the factorization of $\check{D}_j$. One can check that

$$\check{f}_{14}v_\lambda \simeq (f_{13}f_\alpha - f_\alpha f_\beta f_\alpha \bar{q}^3)[\theta+3]_q[\theta+2]_q, \tag{8.17}$$

$$\check{f}_{25}v_\lambda \simeq \left(f_{24}f_\alpha \bar{q}^2\frac{1-q^{2\theta-2}}{q-q^{-1}}+f_\beta f_\alpha^2\frac{1}{[2]}\left(\bar{q}[3]_q+\frac{1-q^{2\theta}}{q-q^{-1}}\right)\right)\delta_5[\theta+2]_q v_\lambda, \tag{8.18}$$

$$\check{f}_{47}v_\lambda \simeq (f_{46}f_\alpha q^3 - f_\alpha f_\beta f_\alpha)[\theta+3]_q[\theta+2]_q, \tag{8.19}$$

in the module $M_\lambda^{\mathfrak{k}}$. Now the proof for $j = 4$ readily follows from Eq. 8.17.

Furthermore, Eq. 8.18 implies that $\check{f}_{15}$ is not divisible by $[\theta]_q$ since $e_\alpha \check{f}_{15} \simeq [3\theta + 3]_q \check{f}_{25}$, and divisible by $[\theta + 2]_q$ by Lemma 6.1. Therefore, $\check{f}_{15}/(\delta_5[\theta + 1]_q[\theta + 2]_q)$ is regular and never turns zero. This proves the case $j = 5$.

Equality (8.19) implies that $\check{f}_{i7}$ are divisible by $[\theta+2]_q[\theta+3]_q$ for all $i \leqslant 4$ by Lemma 6.1. Since $[\theta+2]_q \sqsubset \bar{A}_2^7 \sqsubset \phi_7^{\mathfrak{k}}$, we have $[\theta+2]_q^2 \sqsubset \bar{A}_2^7\check{f}_{37}v_\lambda \simeq e_\beta \check{f}_{27}v_\lambda$ and need to prove $[\theta + 2]_q^2 \sqsubset \check{f}_{27}$. It is easy to check that the operator

$$M_\lambda^{\mathfrak{k}}[\lambda-3\alpha-2\beta]=\mathrm{Span}\{f_\beta f_\alpha^i f_\beta f_\alpha^{3-i}v_\lambda\}_{i=0}^2 \xrightarrow{e_\beta} \mathrm{Span}\{f_\alpha^i f_\beta f_\alpha^{3-i}v_\lambda\}_{i=0}^2=M_\lambda^{\mathfrak{k}}[\lambda-3\alpha-\beta]$$

is independent of λ and has zero kernel. Therefore $\check{f}_{27}$ is divisible by $[\theta + 2]_q^2$ as required. ∎

Define $D_j^{\mathfrak{k}}$ from the equality $\wp_j^{\mathfrak{k}}(u_j^{\mathfrak{k}}) = D_j^{\mathfrak{k}} w_j^\lambda$ and put $D^{\mathfrak{k}} = \prod_{j\in I^{\mathfrak{k}}} D_j^{\mathfrak{k}}$. By construction, $D_j^{\mathfrak{k}} \simeq D_j/(\phi_j^{\mathfrak{k}}\psi_j^{\mathfrak{k}})$.

8.3 Decomposition of $V \otimes M_\lambda^{\mathfrak{k}_l}$

For a long positive root ν, fix $t \in T^\nu$ with the stabilizer $\mathfrak{k} \simeq \mathfrak{k}_l$ and determine $\lambda \in \Theta^{\mathfrak{k}}$ from the equality $q^{2h_\lambda+2h_\rho} = tq^{h_\nu}$. We use $x \in \mathbb{C}^\star$, $x^2, x^3 \neq 1$, to parameterise the spectrum $\{x^{\pm2}, x^{\pm1}, 1\}$ of t as in Eqs. 2.5 and 2.6, for the regular and, respectively, borderline cases. Then the spectrum of the operator $q^{2(\rho,\nu_1)}\mathcal{Q}$ on the module $V \otimes M_\lambda^{\mathfrak{k}_l}$ is $\{x^{\pm2}, q^3x^{\pm1}, q^{-2}\}$.

Proposition 8.2 *The module $V \otimes M_\lambda^{\mathfrak{k}}$ splits into direct sum $\oplus_{j\in I^{\mathfrak{k}}} M_j^{\mathfrak{k}}$ if and only if a) $q^3 \neq x^{\pm1}$, $\frac{q^3-x^{\pm3}}{q-x^{\pm1}} \neq 0$, for $\nu = \beta$, b) $q^3 \neq x^{\pm1}, x^{\pm3}$ for $\nu \neq \beta$.*

Proof Assuming $t \in T^\nu$, we get $D^{\mathfrak{k}} \simeq (x - q^3)^2(x - \bar{q}^3)^2\frac{x^3-q^3}{x-q}\frac{\bar{x}^3-q^3}{\bar{x}-q}$ for $\nu = \beta$ and $D^{\mathfrak{k}} \simeq (x - q^3)^2(x - \bar{q}^3)^2(x^3 - q^3)(\bar{x}^3 - q^3)$ for $\nu \neq \beta$. The sum $\sum_{j\in I^{\mathfrak{k}}} M_j^{\mathfrak{k}}$ exhausts all of $V \otimes M_\lambda^{\mathfrak{k}}$ if and only if $D^{\mathfrak{k}} \neq 0$.

Now suppose that $D^{\mathfrak{k}} \neq 0$. Then the sum $\sum_{j\in I^{\mathfrak{k}}} M_j^{\mathfrak{k}}$ is direct for $\nu = \beta$ by Proposition 5.5. For $\nu \neq \beta$, that is obvious if the $\mathcal{Q}$-eigenvalues are pairwise distinct (which is violated for a finite number of q). Still it is true in all cases. We give a sketch of the proof based on character analysis. One can check that $M_j^{\mathfrak{k}}$ are quotients of $\tilde{M}_j^{\mathfrak{k}}$, where $\tilde{M}_j^{\mathfrak{k}} = M_{\lambda+\nu_j}/M_{\lambda+\nu_j-\ell_j\nu}$ with $\ell_j = \frac{2(\nu_j,\nu)}{(\nu,\nu)} + 1 \in \{1, 2\}$. It is easy to see that

Table 3 Pseudo-parabolic type

$\Pi_{\mathfrak{k}}$	t	$q^{2(\lambda,\alpha)}$	$q^{2(\lambda,\beta)}$	$I^{\mathfrak{k}}$
$\{\beta, 3\alpha+\beta\}$	$(e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}})$	$e^{-\frac{2\pi i}{3}}q^{-2}$	1	$\{1,2,4\}$
$\{\beta, 3\alpha+\beta\}$	$(e^{-\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}})$	$e^{\frac{2\pi i}{3}}q^{-2}$	1	$\{1,2,4\}$
$\{\alpha, 3\alpha+2\beta\}$	$(-1,-1,1)$	1	$-q^{-6}$	$\{1,3\}$
$\{\alpha+\beta, 3\alpha+\beta\}$	$(-1,1,-1)$	-1	$-q^{-6}$	$\{1,2\}$
$\{\beta, 2\alpha+\beta\}$	$(1,-1,-1)$	$-q^{-4}$	1	$\{1,2\}$

$\sum_{j\in I^{\mathfrak{k}}} \mathrm{ch}\tilde{M}^{\mathfrak{k}}_j = \mathrm{ch}V \times \frac{e^{\lambda}(1-e^{-\nu})}{\prod_{\alpha\in\mathrm{R}^+}(1-e^{-\alpha})} = \mathrm{ch}(V\otimes M^{\mathfrak{k}}_{\lambda})$. Now the map $\oplus_{j\in I^{\mathfrak{k}}}\tilde{M}^{\mathfrak{k}}_j \to V\otimes M^{\mathfrak{k}_l}_{\lambda}$ is injective because it is surjective. This also implies $\tilde{M}^{\mathfrak{k}}_j \simeq M^{\mathfrak{k}}_j$ for all $j\in I^{\mathfrak{k}}$. □

9 Decomposition of $V\otimes M^{\mathfrak{k}}_{\lambda}$ for Pseudo-Levi $\mathfrak{k}$

Let μ and ν denote, respectively, the minimal and maximal roots in $\Pi_{\mathfrak{k}}$, relative to the partial order induced from $\mathbb{Z}\mathrm{R}$. Put $\mathfrak{m}\subset\mathfrak{k}$ to be the reductive subalgebra of maximal rank such that $\Pi_{\mathfrak{m}} = \{\mu\}$. For $\lambda\in\Theta^{\mathfrak{k}}\subset\Theta^{\mathfrak{m}}$, the homomorphism $M_{\lambda}\to M^{\mathfrak{k}}_{\lambda}$ factors through the projection $M^{\mathfrak{m}}_{\lambda}\to M^{\mathfrak{k}}_{\lambda}$. Its restriction $M^{\mathfrak{m}}_{\lambda}[\lambda-\xi]\to M^{\mathfrak{k}}_{\lambda}[\lambda-\xi]$ is an isomorphism for $\xi<\nu$. It follows from here that the map $\sum_{\nu_i<\nu} w_i\otimes M^{\mathfrak{m}}_{\lambda}[\lambda-\nu_i]\to\sum_{\nu_i<\nu} w_i\otimes M^{\mathfrak{k}}_{\lambda}[\lambda-\nu_i]$ sends non-vanishing singular vectors $u^{\mathfrak{m}}_j$ with $j\in I^{\mathfrak{k}}\subset I^{\mathfrak{m}}$ over to singular vectors, $u^{\mathfrak{k}}_j\in V\otimes M^{\mathfrak{k}}_{\lambda}$. It follows that $D^{\mathfrak{k}}_j(\lambda) = D^{\mathfrak{m}}_j(\lambda)\neq 0$, for all $j\in I^{\mathfrak{k}}$.

Proposition 9.1 *For all pseudo-Levi $\mathfrak{k}\in\mathfrak{g}$, $V\otimes M^{\mathfrak{k}}_{\lambda} = \oplus_{j\in I^{\mathfrak{k}}} M^{\mathfrak{k}}_j$.*

Proof Since $D^{\mathfrak{k}}\neq 0$, the sum $\sum_{j\in I^{\mathfrak{k}}} M^{\mathfrak{k}}_j$ gives all $V\otimes M^{\mathfrak{k}}_{\lambda}$. It is direct as the $\mathcal{Q}$-eigenvalues are distinct. Indeed, for $\mathfrak{k}=\mathfrak{k}_{l,l}$ we have

$$q^{2\xi_{12}} = e^{\mp\frac{2\pi i}{3}}, \quad q^{2\xi_{14}} = e^{\pm\frac{2\pi i}{3}}q^{8}, \quad q^{2\xi_{24}} = e^{\mp\frac{2\pi i}{3}}q^{6},$$

where the upper sign corresponds to the first row in Table 3. For the three $\mathfrak{k}_{s,l}$-points we have

$$q^{2\xi_{13}} = -q^{2}, \quad q^{2\xi_{12}} = -q^{2}, \quad q^{2\xi_{12}} = -q^{-2},$$

respectively, from the top downward. □

Acknowledgements We are grateful to the anonymous referee for careful reading and valuable remarks and suggestions.

Appendix A

A.1 Formulas for η_{ij} and ξ_{ij}

Below we present the explicit expressions for η_{ij} and ξ_{ij}, $i \leqslant j$, arranging them into matrices.

$$(\eta_{ij}) = \begin{pmatrix} 0 & h_\alpha & h_{\alpha+\beta}+3 & h_{2\alpha+\beta}+4 & h_{3\alpha+\beta}+3 & h_{3\alpha+2\beta}+6 & h_{4\alpha+2\beta}+6 \\ & 0 & h_\beta & h_{\alpha+\beta}+3 & h_{2\alpha+\beta}+4 & h_{2\alpha+2\beta}+4 & h_{3\alpha+2\beta}+6, \\ & & 0 & h_\alpha & h_{2\alpha}-2 & h_{2\alpha+\beta}+4 & h_{3\alpha+\beta}+3 \\ & & & 0 & h_\alpha & h_{\alpha+\beta}+3 & h_{2\alpha+\beta}+4 \\ & & & & 0 & h_\beta & h_{\alpha+\beta}+3 \\ & & & & & 0 & h_\alpha \\ & & & & & & 0 \end{pmatrix}$$

$$(\xi_{ij}) = \begin{pmatrix} 0 & h_\alpha+1 & h_{\alpha+\beta}+4 & h_{2\alpha+\beta}+6 & h_{3\alpha+\beta}+6 & h_{3\alpha+2\beta}+9 & h_{4\alpha+2\beta}+10 \\ & 0 & h_\beta+3 & h_{\alpha+\beta}+5 & h_{2\alpha+\beta}+5 & h_{2\alpha+2\beta}+8 & h_{3\alpha+2\beta}+9, \\ & & 0 & h_\alpha+2 & h_{2\alpha}+2 & h_{2\alpha+\beta}+5 & h_{3\alpha+\beta}+6 \\ & & & 0 & h_\alpha & h_{\alpha+\beta}+3 & h_{2\alpha+\beta}+4 \\ & & & & 0 & h_\beta+3 & h_{\alpha+\beta}+4 \\ & & & & & 0 & h_\alpha+1 \\ & & & & & & 0 \end{pmatrix}$$

A.2 Entries of the Matrix F

Here we present explicit expressions of the entries f_{ij}, $i < j$, participating in the reduced Shapovalov inverse form.

$$f_{12} = f_\alpha, \quad f_{23} = [3]_q f_\beta, \quad f_{34} = f_\alpha, \quad f_{45} = f_\alpha, \quad f_{56} = [3]_q f_\beta, \quad f_{67} = f_\alpha,$$

$$f_{13} = [f_\beta, f_\alpha]_{q^3}, \quad f_{24} = [f_\alpha, f_\beta]_{q^3}, \quad f_{35} = \frac{\bar{q}^2}{[2]_q}[f_\alpha, f_\alpha]_{q^2}, \quad f_{46} = [f_\beta, f_\alpha]_{q^3},$$
$$f_{57} = [f_\alpha, f_\beta]_{q^3},$$

$$f_{14} = \frac{\bar{q}[f_\alpha, [f_\beta, f_\alpha]_{q^3}]_{q^3}}{[2]_q}, \quad f_{25} = \frac{[f_\alpha, [f_\alpha, f_\beta]_q]_{q^3}}{[2]_q^2}, \quad f_{36} = \frac{[[f_\beta, f_\alpha]_q, f_\alpha]_{q^3}}{[2]_q^2},$$
$$f_{47} = \frac{\bar{q}[[f_\alpha, f_\beta]_{q^3}, f_\alpha]_{q^3}}{[2]_q}$$

$$f_{15} = \frac{\bar{q}^2[f_\alpha, [f_\alpha, [f_\beta, f_\alpha]_{q^3}]_{q^3}]_q}{[2]_q^2} \quad f_{26} = \frac{\bar{q}^3[f_\beta, [f_\alpha, [f_\alpha, f_\beta]_q]_{q^3}]_{q^6}}{[2]_q^2},$$
$$f_{37} = \frac{\bar{q}^2[[[f_\alpha, f_\beta]_{q^3}, f_\alpha]_{q^3}, f_\alpha]_q}{[2]_q^2},$$

$$f_{16} = \frac{\bar{q}^2}{[2]_q^2}[f_\beta, [f_\alpha, [f_\alpha, [f_\beta, f_\alpha]_{q^3}]_{q^3}]_q]_{q^3}, \quad f_{27} = \frac{\bar{q}^2}{[2]_q^2}[[[[f_\alpha, f_\beta]_{q^3}, f_\alpha]_{q^3}, f_\alpha]_q, f_\beta]_{q^3},$$
$$f_{17} = \frac{\bar{q}^2}{[2]_q^2}[f_\alpha, [f_\beta, [f_\alpha, [f_\alpha, [f_\beta, f_\alpha]_{q^3}]_{q^3}]_q]_{q^3}]_q.$$

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